MODULI SPACE AND STRUCTURE OF NONCOMMUTATIVE 3-SPHERES

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Abstract

We analyse the moduli space and the structure of noncommutative 3-spheres. We develop the notion of central quadratic form for quadratic algebras, and prove a general algebraic result which considerably refines the classical homomorphism from a quadratic algebra to a cross-product algebra associated to the characteristic variety and lands in a richer cross-product. It allows to control the C^*-norm on involutive quadratic algebras and to construct the differential calculus in the desired generality. The moduli space of noncommutative 3-spheres is identified with equivalence classes of pairs of points in a symmetric space of unitary unimodular symmetric matrices. The scaling foliation of the moduli space is identified to the gradient flow of the character of a virtual representation of $SO(6)$. Its generic orbits are connected components of real parts of elliptic curves which form a net of biquadratic curves with 8 points in common. We show that generically these curves are the same as the characteristic variety of the associated quadratic algebra. We then apply the general theory of central quadratic forms to show that the noncommutative 3-spheres admit a natural ramified covering π by a noncommutative 3-dimensional nilmanifold. This yields the differential calculus. We then compute the Jacobian of the ramified covering π by pairing the direct image of the fundamental class of the noncommutative 3–dimensional nilmanifold with the Chern character of the defining unitary and obtain the answer as the product of a period (of an elliptic integral) by a rational function. Finally we show that the hyperfinite factor of type II_1 appears as cross-product of the field K_q of meromorphic functions on an elliptic curve by a subgroup of its Galois group $\mathrm{Aut}_{\mathbb{C}}(K_q)$.

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1 Introduction

Noncommutative Differential Geometry is a growing subject centering around the exploration of a new kind of geometric spaces which do not belong to the classical geometric world. The theory is resting on large classes of examples as well as on the elaboration of new general concepts. The main sources of examples so far have been provided by:

1) A general principle allowing to understand difficult quotients of classical spaces, typically spaces of leaves of foliations, as noncommutative spaces.

2) Deformation theory which provides rich sources of examples in particular in the context of "quantization" problems.

3) Spaces of direct relevance in physics such as the Brillouin zone in the quantum Hall effect, or even space-time as in the context of the standard model with its Higgs sector.

We recently came across a whole class of new noncommutative spaces defined as solutions of a basic equation of K-theoretic origin. This equation was at first expected to admit only commutative (or nearly commutative) solutions. Whereas classical spheres provide simple solutions of arbitrary dimension d, it turned out that when the dimension d is ≥ 3 there are very interesting new, and highly noncommutative, solutions. The first examples were given in [10], and in [9] (hereafter refered to as Part I) we began the classification of all solutions in the 3-dimensional case, by giving an exhaustive list of noncommutative 3-spheres $S^3_{\mathbf{u}}$, and analysing the "critical" cases. We also explained in [9] a basic relation, for generic values of our "modulus" $\mathbf{u}$, between the algebra of coordinates on the noncommutative 4-space of which $S^3_{\mathbf{u}}$ is the unit sphere, and the Sklyanin algebras which were introduced in the context of totally integrable systems.

In this work we analyse the structure of noncommutative 3-spheres $S^3_{\mathbf{u}}$ and of their moduli space. We started from the above relation with the Sklyanin algebras and first computed basic cyclic cohomology invariants using θ-functions. The invariant to be computed was depending on an elliptic curve (with modulus $q = e^{\pi i \tau}$) and several points on the curve. It appeared as a sum of 1440 terms, each an integral over a period of a rational fraction of high degree (16) in θ-functions and their derivatives. After computing the first terms in the q-expansion of the sum, (with the help of a computer[3]), and factoring out basic elliptic functions of the above parameters, we were left with a scalar

[3] We wish to express our gratitude to Michael Trott for his kind assistance

function of q, starting as,

$$q^{3/4} - 9\,q^{11/4} + 27\,q^{19/4} - 12\,q^{27/4} - 90\,q^{35/4} + \ldots$$

in which one recognises the 9^{th} power of the Dedekind η-function.

We then gradually simplified the result (with η^9 appearing as an integration factor from the derivative of the Weierstrass $\wp$-function) and elaborated the concepts which directly explain the final form of the result.

The main new conceptual tool which we obtained and that we develop in this paper is the notion of central quadratic form for quadratic algebras (Definition 6). The geometric data $\{E\,,\,\sigma\,,\,\mathcal{L}\}$ of a quadratic algebra $\mathcal{A}$ is a standard notion ([14], [3], [19]) defined in such a way that the algebra maps homomorphically to a cross-product algebra obtained from sections of powers of the line bundle $\mathcal{L}$ on powers of the correspondence σ. We shall prove a purely algebraic result (Lemma 7) which considerably refines the above homomorphism and lands in a richer cross-product. Its origin can be traced as explained above to the work of Sklyanin ([17]) and Odesskii-Feigin ([14]).

Our construction is then refined to control the C^*-norm (Theorem 10) and to construct the differential calculus in the desired generality (section 8). It also allows to show the pertinence of the general “spectral" framework for noncommutative geometry, in spite of the rather esoteric nature of the above examples.

We apply the general theory of central quadratic forms to show that the noncommutative 3-spheres $S^3_{\mathbf{u}}$ admit a natural ramified covering π by a non-

commutative 3-manifold M which is (in the even case, cf. Proposition 12) isomorphic to the mapping torus of an outer automorphism of the noncommutative 2-torus T^2_η. It is a noncommutative version of a nilmanifold (Corollary 13) with a natural action of the Heisenberg Lie algebra $\mathfrak{h}_3$ and an invariant trace. This covering yields the differential calculus in its "transcendental" form.

Another important novel concept which plays a basic role in the present paper is the notion of Jacobian for a morphism of noncommutative spaces, developed from basic ideas of noncommutative differential geometry [5] and expressed in terms of Hochschild homology (section 7). We compute the Jacobian of the above ramified covering π by pairing the direct image of the fundamental class of the noncommutative 3-dimensional nilmanifold M with the Chern character of the defining unitary and obtain the answer as the product of a period (of an elliptic integral) by a rational function (Theorem 15 and Corollary 16). As explained above, we first computed these expressions in terms of elliptic functions and modular forms which led us in order to simplify the results to extend the moduli space from the real to the complex domain, and to formulate everything in geometric terms.

The leaves of the scaling foliation of the real moduli space then appear as the real parts of a net of degree 4 elliptic curves in $P_3(\mathbb{C})$ having 8 points in common. These elliptic curves turn out to play a fundamental role and to be closely related to the elliptic curves of the geometric data (cf. Section 4) of the quadratic algebras which their elements label. In fact we first directly compared the j-invariant of the generic fibers (of the scaling foliation) with the j-invariant of the quadratic algebras, and found them to be equal. This equality is surprising in that it fails in the degenerate (non-generic) cases, where the characteristic variety can be as large as $P_3(\mathbb{C})$. Moreover, even in

the generic case, the two notions of "real" points, either in the fiber or in the characteristic variety are not the same, but dual to each other. We eventually explain in Theorem 5 the generic coincidence between the leaves of the scaling foliation in the complexified moduli space and the characteristic varieties of the associated quadratic algebras. This Theorem 5 also exhibits the relation of our theory with iterations of a specific birational automorphism of $P_3(\mathbb{C})$, defined over $\mathbb{Q}$, and restricting on each fiber as a translation of this elliptic curve. The generic irrationality of this translation and the nature of its diophantine approximation play an important role in sections 7 and 8.

Another important result is the appearance of the hyperfinite factor of type II_1 as cross-product of the field K_q of meromorphic functions on an elliptic curve by a subgroup of the Galois group $\mathrm{Aut}_{\mathbb{C}}(K_q)$, (Theorem 14) and the description of the differential calculus in general (Lemma 19) and in "rational" form on $S^3_{\mathbf{u}}$ (Theorem 21). The detailed proofs together with the analysis of the spectral geometry of $S^3_{\mathbf{u}}$ and of the C^*-algebra $C^*(S^3_{\mathbf{u}})$ will appear in Part II.

2 The Real Moduli Space of 3-Spheres $S^3_{\mathbf{u}}$

Let us now be more specific and describe the basic K-theoretic equation defining our spheres. In the simplest case it asserts that the algebra $\mathcal{A}$ of "coordinates" on the noncommutative space is generated by a self-adjoint idempotent e, ($e^2 = e$, $e = e^*$) together with the algebra $M_2(\mathbb{C})$ of two by two scalar matrices. The only relation is that the trace of e vanishes, i.e. that the projection of e on the commutant of $M_2(\mathbb{C})$ is zero. One shows that $\mathcal{A}$ is then the algebra $M_2(C_{\mathrm{alg}}(S^2))$ where $C_{\mathrm{alg}}(S^2)$ is the algebra of coordinates

on the standard 2-sphere S^2.

The general form of the equation distinguishes two cases according to the parity of the dimension d. In the even case the algebra $\mathcal{A}$ of “coordinates" on the noncommutative space is still generated by a projection e, ($e^2 = e$, $e = e^*$) and an algebra of scalar matrices, but the dimension $d = 2k$ appears in requiring the vanishing not only of the trace of e, but of all components of its Chern character of degree $0, \ldots, d-2$. This of course involves the algebraic (cyclic homology) formulation of the Chern Character.
The Chern character in cyclic homology [5], [7],

$$\mathrm{ch}_* : \quad \mathrm{K}_*(\mathcal{A}) \to \mathrm{HC}_*(\mathcal{A}) \tag{2.1}$$

is the noncommutative geometric analogue of the classical Chern character. We describe it in the odd case which is relevant in our case $d = 3$. Given a noncommutative algebra $\mathcal{A}$, an invertible element U in $M_d(\mathcal{A})$ defines a class in $K_1(\mathcal{A})$ and the components of its Chern character are given by,

$$\begin{aligned} \mathrm{ch}_{\frac{n}{2}}(U) &= U^{i_0}_{i_1} \otimes V^{i_1}_{i_2} \otimes \cdots \otimes U^{i_{n-1}}_{i_n} \otimes V^{i_n}_{i_0} \\ &- V^{i_0}_{i_1} \otimes U^{i_1}_{i_2} \otimes \cdots \otimes V^{i_{n-1}}_{i_n} \otimes U^{i_n}_{i_0} \end{aligned} \tag{2.2}$$

where $V := U^{-1}$ and summation over repeated indices is understood.
By a noncommutative n-dimensional spherical manifold (n odd), we mean the noncommutative space S dual to the $*$-algebra $\mathcal{A}$ generated by the components U^i_j of a unitary solution $U \in M_d(\mathcal{A})$, $d = 2^{\frac{n-1}{2}}$, of the equation

$$\mathrm{ch}_{\frac{k}{2}}(U) = 0\,, \quad \forall k < n\,,\ k\,\mathrm{odd}\,, \quad \mathrm{ch}_{\frac{n}{2}}(U) \neq 0 \tag{2.3}$$

which is the noncommutative counterpart of the vanishing of the lower Chern classes of the Bott generator of the K-theory of classical odd spheres.

The moduli space of 3-dimensional spherical manifolds appears naturally as a quotient of the space of symmetric unitary matrices

$$\mathcal{S} := \{ \Lambda \in M_4(\mathbb{C}) \mid \quad \Lambda = \Lambda^t , \quad \Lambda^* = \Lambda^{-1} \} \tag{2.4}$$

Indeed for any $\Lambda \in \mathcal{S}$ let $U(\Lambda)$ be the unitary solution of (2.3) given by

$$U = \mathbb{1}_2 \otimes z^0 + i\sigma_k \otimes z^k \tag{2.5}$$

where σ_k are the usual Pauli matrices and the presentation of the involutive algebra $C_{\mathrm{alg}}(S^3(\Lambda))$ generated by the z^μ is given by the relations

$$U^* U \;=\; U\, U^* \;=\; 1 \,, \quad z^{\mu *} = \Lambda^\mu_\nu z^\nu \tag{2.6}$$

We let $\mathcal{A} := C_{\mathrm{alg}}(\mathbb{R}^4(\Lambda))$ be the associated quadratic algebra, generated by the z^μ with presentation,

$$U^* U \;=\; U\, U^* \;\in \mathbb{1}_2 \otimes \mathcal{A} \,, \quad z^{\mu *} = \Lambda^\mu_\nu z^\nu \tag{2.7}$$

where $\mathbb{1}_2$ is the unit of $M_2(\mathbb{C})$. The element $r^2 := \sum_{\mu=0}^3 z^\mu z^{\mu *} = \sum_{\mu=0}^3 z^{\mu *} z^\mu$ is in the center of $C_{\mathrm{alg}}(\mathbb{R}^4(\Lambda))$ (cf. Part I) and the additional inhomogeneous relation defining $C_{\mathrm{alg}}(S^3(\Lambda))$ is $r^2 = 1$.

By Part I Theorem 1, any unitary solution of (2.3) for $n = 3$ is a homomorphic image of $U(\Lambda)$ for some $\Lambda \in \mathcal{S}$. The transformations

$$U \mapsto \lambda\, U \;\text{ with }\; \lambda = e^{i\varphi} \in U(1) \tag{2.8}$$

$$U \mapsto V_1\, U\, V_2 \;\text{ with }\; V_1, V_2 \in SU(2) \tag{2.9}$$

$$U \mapsto U^* \tag{2.10}$$

act on the space of unitary solutions of (2.3) and preserve the isomorphism class of the algebra $C_{\mathrm{alg}}(S^3(\Lambda))$ and of the associated quadratic algebra $C_{\mathrm{alg}}(\mathbb{R}^4(\Lambda))$. Transformation (2.8) corresponds to

$$\Lambda \mapsto e^{-2i\varphi}\,\Lambda \tag{2.11}$$

Transformation (2.9) induces $z^\mu \mapsto S^\mu_\nu z^\nu$ with $S \in SO(4)$ which in turn induces

$$\Lambda \mapsto S\,\Lambda\,S^t\,, \quad S \in SO(4) \tag{2.12}$$

Finally (2.10) reverses the "orientation" of $S^3(\Lambda)$ and corresponds to

$$\Lambda \mapsto \Lambda^{-1} \tag{2.13}$$

We define the *real moduli space* $\mathcal{M}$ as the quotient of $\mathcal{S}$ by the transformations (2.11) and (2.12), and the *unoriented* real moduli space $\mathcal{M}'$ as its quotient by (2.13).

By construction the space $\mathcal{S}$ is the homogeneous space $U(4)/O(4)$, with $U(4)$ acting on $\mathcal{S}$ by

$$\Lambda \mapsto V\Lambda V^t \tag{2.14}$$

for $\Lambda \in \mathcal{S}$ and $V \in U(4)$. The conceptual description of the moduli spaces $\mathcal{M}$ and $\mathcal{M}'$ requires taking care of finer details. We let θ be the involution of the compact Lie group $SU(4)$ given by complex conjugation, and define the closed subgroup $K \subset SU(4)$ as the normaliser of $SO(4) \subset SU(4)$, i.e. by the condition

$$K := \{u \in SU(4)\,|\,u^{-1}\,\theta(u)\,\in\,Z\} \tag{2.15}$$

where Z is the center of $SU(4)$. The quotient $X := SU(4)/K$ is a Riemannian globally symmetric space (cf. [13] Theorem 9.1, Chapter VII). One has $Z \subset K$ but the image of K in $U := SU(4)/Z$ is disconnected. Indeed, besides $Z \cdot SO(4)$ the subgroup K contains the diagonal matrices with $\{v, v, v, v^{-3}\}$ as diagonal elements, where v is an 8th root of 1.

Proposition 1 *The real moduli space $\mathcal{M}$ (resp. $\mathcal{M}'$) is canonically isomorphic to the space of congruence classes of point pairs under the action of $SU(4)$ (resp. of isometries) in the Riemannian globally symmetric space $X = SU(4)/K$.*

This gives two equivalent descriptions of $\mathcal{M}$ as the orbifold quotient of a 3-torus by the action of the Weyl group of the symmetric pair.
In the first (A_3) we identify the Lie algebra $\mathfrak{su}(4) = \mathrm{Lie}(SU(4))$ with the Lie algebra of traceless antihermitian elements of $M_4(\mathbb{C})$. We let $\mathfrak{d}$ be the Lie subalgebra of diagonal matrices, it is a maximal abelian subspace of $\mathfrak{su}(4)_- = \{X \in \mathfrak{su}(4) | \theta(X) = -X\}$. The Weyl group W of the symmetric pair $(SU(4), K)$ is isomorphic to the permutation group $\mathfrak{S}_4$ acting on $\mathfrak{d}$ by permutation of the matrix elements. The 3-torus T_A is the quotient,

$$T_A := \mathfrak{d}\,/\Gamma\,, \quad \Gamma = \{\delta \in \mathfrak{d} | e^{\delta} \in K\} \tag{2.16}$$

and the isomorphism of $T_A\,/W$ with $\mathcal{M}$ is obtained from,

$$\sigma(\delta) := e^{2\delta} \in \mathcal{S} \quad \forall \delta \in \mathfrak{d} \tag{2.17}$$

The lattice Γ is best expressed as $\Gamma = \{\delta \in \mathfrak{d} | \langle \delta, \rho \rangle \in \pi i\,\mathbb{Z},\ \forall \rho \in \Delta\}$ in terms of the roots ρ of the pair $(SU(4), K)$ where the root system Δ is the same as for the Cartan subalgebra $\mathfrak{d}_{\mathbb{C}} = \mathfrak{d} \otimes \mathbb{C} \subset \mathfrak{sl}(4, \mathbb{C})$ of $\mathfrak{sl}(4, \mathbb{C}) = \mathfrak{su}(4)_{\mathbb{C}}$. The roots $\alpha_{\mu,\nu}$, $\mu, \nu \in \{0, 1, 2, 3\}$, $\mu \neq \nu$ are given by

$$\alpha_{\mu,\nu}(\delta) = \alpha_{\mu}(\delta) - \alpha_{\mu}(\delta) \tag{2.18}$$

where $\alpha_{\mu}(\delta)$ for $\mu \in \{0, 1, 2, 3\}$ are the elements of the diagonal matrix $\delta \in \mathfrak{d}$. In terms of the primitive roots $\alpha_{0,k}$, $k \in \{1, 2, 3\}$ the coordinates φ_k used in Part I are given by $\varphi_k = \frac{1}{i}\alpha_{0,k}$ and the lattice Γ corresponds to $\varphi_k \in$

$\pi\mathbb{Z}$, $\forall k \in \{1,2,3\}$. With $T := \mathbb{R}/\pi\mathbb{Z}$ we let $\delta : T^3 \mapsto T_A$ be the inverse isomorphism. One has, modulo projective equivalence,

$$\sigma(\delta_\varphi) \sim \begin{bmatrix} 1 & & & \\ & e^{-2i\varphi_1} & & \\ & & e^{-2i\varphi_2} & \\ & & & e^{-2i\varphi_3} \end{bmatrix} \tag{2.19}$$

In these coordinates (φ_k) the symmetry given by the Weyl group $W = \mathfrak{S}_4$ of the symmetric pair $(SU(4), K)$ now reads as follows,

$$\begin{aligned} T_{01}(\varphi_1,\varphi_2,\varphi_3) &= (-\varphi_1, \varphi_2 - \varphi_1, \varphi_3 - \varphi_1) \\ T_{12}(\varphi_1,\varphi_2,\varphi_3) &= (\varphi_2,\varphi_1,\varphi_3) \\ T_{23}(\varphi_1,\varphi_2,\varphi_3) &= (\varphi_1,\varphi_3,\varphi_2) \end{aligned} \tag{2.20}$$

where $T_{\mu\nu}$ is the transposition of $\mu, \nu \in \{0,1,2,3\}$, $\mu < \nu$.

In Part I we used the parametrization by $\mathbf{u} \in T^3$ to label the 3-dimensional spherical manifolds $S^3_{\mathbf{u}}$ and their quadratic counterparts $\mathbb{R}^4_{\mathbf{u}}$. The presentation of the algebra $C_{\rm alg}(\mathbb{R}^4_{\mathbf{u}})$ is given as follows, using the selfadjoint generators $x^\mu = x^{\mu *}$ for $\mu \in \{0,1,2,3\}$ related to the z^k by $z^0 = x^0$, $z^k = e^{i\varphi_k}\, x^k$ for $j \in \{1,2,3\}$

$$\cos(\varphi_k)[x^0, x^k]_- = i\, \sin(\varphi_\ell - \varphi_m)[x^\ell, x^m]_+ \tag{2.21}$$

$$\cos(\varphi_\ell - \varphi_m)[x^\ell, x^m]_- = -i\, \sin(\varphi_k)[x^0, x^k]_+ \tag{2.22}$$

for $k = 1,2,3$ where (k,ℓ,m) is the cyclic permutation of $(1,2,3)$ starting with k and where $[a,b]_\pm = ab \pm ba$. The algebra $C_{\rm alg}(S^3_{\mathbf{u}})$ is the quotient of $C_{\rm alg}(\mathbb{R}^4_{\mathbf{u}})$ by the two-sided ideal generated by the hermitian central element $\sum_\mu (x^\mu)^2 - \mathbb{1}$.

The second equivalent description of $\mathcal{M}$ relies on the equality $A_3 = D_3$, i.e. the identification of $SU(4)$ with the Spin covering of $SO(6)$ using the

(4-dimensional) half spin representation of the latter. Proposition 1 holds unchanged replacing the pair $(SU(4), K)$ by the pair $(SO(6), K_D)$, where K_D corresponds under the above isomorphism with the quotient of K by the kernel of the covering $\mathrm{Spin}(6) \mapsto SO(6)$. Identifying the Lie algebra $\mathfrak{so}(6)$ with the Lie algebra of antisymmetric six by six real matrices, $\mathfrak{d}$ corresponds to the subalgebra $\mathfrak{d}_D$ of block diagonal matrices with three blocks of the form

$$\begin{pmatrix} 0 & \psi_k \\ -\psi_k & 0 \end{pmatrix} \tag{2.23}$$

The 3-torus T_D is the quotient,

$$T_D := \mathfrak{d}_D / \Gamma_D , \quad \Gamma_D = \{\delta \in \mathfrak{d}_D \,|\, e^{\delta} \in K_D\} \tag{2.24}$$

which using the roots of D_3, $(\pm e_i \pm e_j)(\psi) := \pm \psi_i \pm \psi_j$ becomes,

$$\Gamma_D = \{\psi \,|\, \psi_i \pm \psi_j \in \pi\mathbb{Z}\} \tag{2.25}$$

The relation between the parameters (ψ_k) for $k \in \{1, 2, 3\}$ and the (φ_k) is given by,

$$2\,\psi_1 = \varphi_2 + \varphi_3 - \varphi_1, \;\; 2\,\psi_2 = \varphi_3 + \varphi_1 - \varphi_2, \;\; 2\,\psi_3 = \varphi_1 + \varphi_2 - \varphi_3 \tag{2.26}$$

and in terms of the ψ_k the Killing metric on T_D reads,

$$ds^2 = (d\psi_1)^2 + (d\psi_2)^2 + (d\psi_3)^2 \tag{2.27}$$

The action of the Weyl group W is given by the subgroup of $O(3, \mathbb{Z})$

$$w(\psi_1, \psi_2, \psi_3) = (\varepsilon_1 \psi_{\sigma(1)}, \varepsilon_2 \psi_{\sigma(2)}, \varepsilon_3 \psi_{\sigma(3)}) \tag{2.28}$$

with $\sigma \in \mathfrak{S}_3$ and $\varepsilon_k \in \{1, -1\}$, of elements $w \in O(3, \mathbb{Z})$ such that $\varepsilon_1 \varepsilon_2 \varepsilon_3 = 1$. The additional symmetry (2.13) defining $\mathcal{M}'$ is simply

$$\psi \mapsto -\psi \tag{2.29}$$

and together with W it generates $O(3,\mathbb{Z})$.

Another advantage of the variable ψ_k is that the scaling vector field Z (cf. Part I) whose orbits describe the local equivalence relation on $\mathcal{M}$ generated by the isomorphism of quadratic algebras,

$$C_{\rm alg}(\mathbb{R}^4(\Lambda_1)) \sim C_{\rm alg}(\mathbb{R}^4(\Lambda_2)) \tag{2.30}$$

and which was given in the variables (φ_k) as,

$$Z = \sum_{k=1}^{3} \sin(2\varphi_k)\sin(\varphi_\ell + \varphi_m - \varphi_k)\frac{\partial}{\partial\varphi_k} \tag{2.31}$$

is now given by

$$Z = \frac{1}{4}\sum_{k=1}^{3} \frac{\partial H_0}{\partial\psi_k}\frac{\partial}{\partial\psi_k} \tag{2.32}$$

with

$$H_0 = \sin(2\,\psi_1)\sin(2\,\psi_2)\sin(2\,\psi_3)\,. \tag{2.33}$$

Since $2\,\Gamma_D$ is the unit lattice for $PSO(6)$ we can translate everything to the space $\mathcal{C}$ of conjugacy classes in $PSO(6)$.

Theorem 2 *The doubling map $\psi \mapsto 2\psi$ establishes an isomorphism between $\mathcal{M}$ and $\mathcal{C}$ transforming the scaling foliation on $\mathcal{M}$ into the gradient flow (for the Killing metric) of the character of the signature representation of $SO(6)$, i.e. the super-trace of its action on $\wedge^3\mathbb{C}^6 = \wedge^3_+\mathbb{C}^6 \oplus \wedge^3_-\mathbb{C}^6$ with $\wedge^3_\pm\mathbb{C}^6 = \{\omega \in \wedge^3\mathbb{C}^6 \mid *\,\omega = \pm\, i\omega\}$.*

This flow admits remarkable compatibility properties with the canonical cell decomposition of $\mathcal{C}$, they will be analysed in Part II.

3 The Complex Moduli Space and its Net of Elliptic Curves

The proper understanding of the noncommutative spheres $S^3_{\mathbf{u}}$ relies on basic computations (Theorem 15 below) whose result depends on $\mathbf{u}$ through elliptic integrals. The conceptual explanation of this dependence requires extending the moduli space from the real to the complex domain. The leaves of the scaling foliation then appear as the real parts of a net of degree 4 elliptic curves in $P_3(\mathbb{C})$ having 8 points in common. These elliptic curves will turn out to play a fundamental role and to be closely related to the elliptic curves of the geometric data (cf. Section 4) of the quadratic algebras which their elements label.

To extend the moduli space to the complex domain we start with the relations defining the involutive algebra $C_{\mathrm{alg}}(S^3(\Lambda))$ and take for Λ the diagonal matrix with

$$\Lambda^\mu_\mu := u_\mu^{-1} \tag{3.1}$$

where (u_0, u_1, u_2, u_3) are the coordinates of $\mathbf{u} \in P_3(\mathbb{C})$. Using $y_\mu := \Lambda^\mu_\nu z^\nu$ one obtains the homogeneous defining relations in the form,

$$\begin{aligned} u_k\, y_k\, y_0 - u_0\, y_0\, y_k + u_\ell\, y_\ell\, y_m - u_m\, y_m\, y_\ell &= 0 \\ u_k\, y_0\, y_k - u_0\, y_k\, y_0 + u_m\, y_\ell\, y_m - u_\ell\, y_m\, y_\ell &= 0 \end{aligned} \tag{3.2}$$

for any cyclic permutation (k, ℓ, m) of (1,2,3). The inhomogeneous relation becomes,

$$\sum u_\mu\, y_\mu^2 = 1 \tag{3.3}$$

and the corresponding algebra $C_{\mathrm{alg}}(S^3_{\mathbb{C}}(\mathbf{u}))$ only depends upon the class of $\mathbf{u} \in P_3(\mathbb{C})$. We let $C_{\mathrm{alg}}(\mathbb{C}^4(\mathbf{u}))$ be the quadratic algebra defined by (3.2).

Taking $u_\mu = e^{2i\,\varphi_\mu}$, $\varphi_0 = 0$, for all μ and $x_\mu := e^{i\,\varphi_\mu} y_\mu$ we obtain the defining relations (2.21) and (2.22) (except for $x_\mu^* = x_\mu$).

We showed in Part I that for $\varphi_k \neq 0$ and $|\varphi_r - \varphi_s| \neq \frac{\pi}{2}$ (i.e. $u_k \neq 1$ and $u_r \neq -u_s$) for any $k, r, s \in \{1, 2, 3\}$, one can find 4 scalars s^μ such that by setting $S_\mu = s^\mu x^\mu$ for $\mu \in \{0, 1, 2, 3\}$ the system (2.21), (2.22) reads

$$[S_0, S_k]_- = i J_{\ell m} [S_\ell, S_m]_+ \tag{3.4}$$

$$[S_\ell, S_m]_- = i [S_0, S_k]_+ \tag{3.5}$$

for $k = 1, 2, 3$, where (k, ℓ, m) is the cyclic permutation of (1,2,3) starting with k and where $J_{\ell m} = -\tan(\varphi_\ell - \varphi_m)\tan(\varphi_k)$. One has

$$J_{12} + J_{23} + J_{31} + J_{12}J_{23}J_{31} = 0 \tag{3.6}$$

and the relations (3.4), (3.5) together with (3.6) for the scalars $J_{\ell m}$ characterize the Sklyanin algebra [17], [18], a regular algebra of global dimension 4 which has been widely studied (see e.g. [14], [19], [20]) and which plays an important role in noncommutative algebraic geometry. From (3.6) it follows that for the above (generic) values of $\mathbf{u}$, $C_{\mathrm{alg}}(\mathbb{R}^4_{\mathbf{u}})$ only depends on 2 parameters; with Z as in (2.31) one has $Z(J_{k\ell}) = 0$ and the leaf of the scaling foliation through a generic $\mathbf{u} \in T^3$ is the connected component of $\mathbf{u}$ in

$$F_{\mathbb{T}}(\mathbf{u}) := \{\mathbf{v} \in T^3 \mid J_{k\ell}(\mathbf{v}) = J_{k\ell}(\mathbf{u})\} \tag{3.7}$$

In terms of homogeneous parameters the functions $J_{\ell m}$ read as

$$J_{\ell m} = \tan(\varphi_0 - \varphi_k)\tan(\varphi_\ell - \varphi_m) \tag{3.8}$$

for any cyclic permutation (k, ℓ, m) of (1,2,3), and extend to the complex domain $\mathbf{u} \in P_3(\mathbb{C})$ as,

$$J_{\ell m} = \frac{(u_0 + u_\ell)(u_m + u_k) - (u_0 + u_m)(u_k + u_\ell)}{(u_0 + u_k)(u_\ell + u_m)} \tag{3.9}$$

It follows easily from the finer Theorem 5 that for generic values of $\mathbf{u} \in P_3(\mathbb{C})$ the quadratic algebra $C_{\text{alg}}(\mathbb{C}^4(\mathbf{u}))$ only depends upon $J_{k\ell}(\mathbf{u})$. We thus define

$$F(\mathbf{u}) := \{\mathbf{v} \in P_3(\mathbb{C}) \,|\, J_{k\ell}(\mathbf{v}) = J_{k\ell}(\mathbf{u})\} \tag{3.10}$$

Let then,

$$(\alpha, \beta, \gamma) = \{(u_0 + u_1)(u_2 + u_3), (u_0 + u_2)(u_3 + u_1), (u_0 + u_3)(u_1 + u_2)\}$$

be the Lagrange resolvent of the 4th degree equation,

$$\Phi(\mathbf{u}) = (\alpha, \beta, \gamma) \tag{3.11}$$

viewed as a map

$$\Phi : P_3(\mathbb{C}) \backslash S \to P_2(\mathbb{C}) \tag{3.12}$$

where S is the following set of 8 points

$$p_0 = (1,0,0,0),\ p_1 = (0,1,0,0),\ p_2 = (0,0,1,0),\ p_3 = (0,0,0,1) \tag{3.13}$$
$$q_0 = (-1,1,1,1),\ q_1 = (1,-1,1,1),\ q_2 = (1,1,-1,1),\ q_3 = (1,1,1,-1)$$

We extend the generic definition (3.10) to arbitray $\mathbf{u} \in P_3(\mathbb{C}) \backslash S$ and define $F_{\mathbf{u}}$ in general as the union of S with the fiber of Φ through $\mathbf{u}$. It can be understood geometrically as follows.

Let $\mathcal{N}$ be the net of quadrics in $P_3(\mathbb{C})$ which contain S. Given $\mathbf{u} \in P_3(\mathbb{C}) \backslash S$ the elements of $\mathcal{N}$ which contain $\mathbf{u}$ form a pencil of quadrics with base locus

$$\cap\{Q \,|\, Q \in \mathcal{N},\ \mathbf{u} \in Q\} = Y_{\mathbf{u}} \tag{3.14}$$

which is an elliptic curve of degree 4 containing S and $\mathbf{u}$. One has

$$Y_{\mathbf{u}} = F_{\mathbf{u}} \tag{3.15}$$

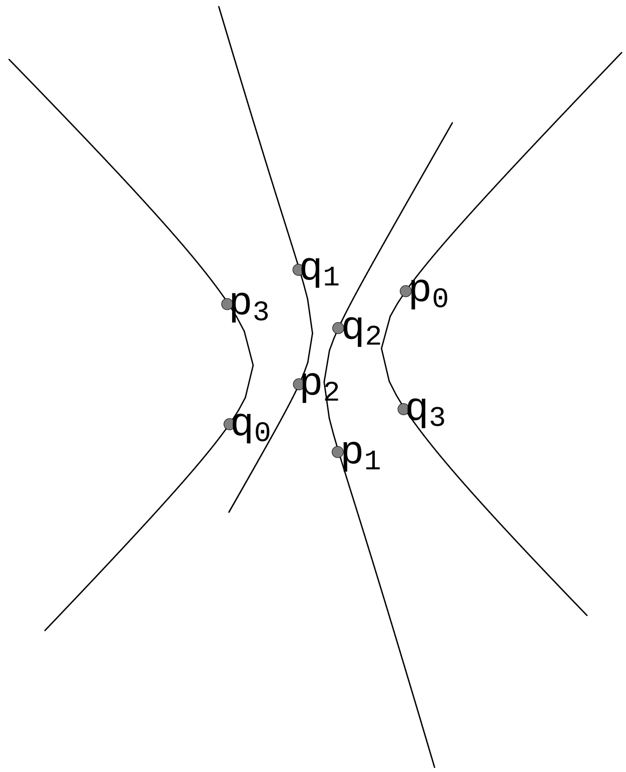

Figure 1: The Elliptic Curve $F_{\mathbf{u}} \cap P_3(\mathbb{R})$

We shall now give, for generic values of (α, β, γ) a parametrization of $F_{\mathbf{u}}$ by θ-functions. We start with the equations for $F_{\mathbf{u}}$

$$\frac{(u_0+u_1)(u_2+u_3)}{\alpha} = \frac{(u_0+u_2)(u_3+u_1)}{\beta} = \frac{(u_0+u_3)(u_1+u_2)}{\gamma} \tag{3.16}$$

and we diagonalize the above quadratic forms as follows

$$\begin{aligned}(u_0+u_1)(u_2+u_3) &= Z_0^2 - Z_1^2 \\ (u_0+u_2)(u_3+u_1) &= Z_0^2 - Z_2^2 \\ (u_0+u_3)(u_1+u_2) &= Z_0^2 - Z_3^2\end{aligned} \tag{3.17}$$

where

$$(Z_0, Z_1, Z_2, Z_3) = M.u \tag{3.18}$$

where M is the involution,

$$M := \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{3.19}$$

In these terms the equations for $F_{\mathbf{u}}$ read

$$\frac{Z_0^2 - Z_1^2}{\alpha} = \frac{Z_0^2 - Z_2^2}{\beta} = \frac{Z_0^2 - Z_3^2}{\gamma} \tag{3.20}$$

Let now $\omega \in \mathbb{C}$, $\operatorname{Im}\omega > 0$ and $\eta \in \mathbb{C}$ be such that one has, modulo projective equivalence,

$$(\alpha, \beta, \gamma) \sim \left(\frac{\theta_2(0)^2}{\theta_2(\eta)^2}, \frac{\theta_3(0)^2}{\theta_3(\eta)^2}, \frac{\theta_4(0)^2}{\theta_4(\eta)^2}\right) \tag{3.21}$$

where $\theta_1, \theta_2, \theta_3, \theta_4$ are the theta functions for the lattice $L = \mathbb{Z} + \mathbb{Z}\omega \subset \mathbb{C}$

Proposition 3 *The following define isomorphisms of $\mathbb{C}/L$ with $F_{\mathbf{u}}$,*

$$\varphi(z) = \left(\frac{\theta_1(2z)}{\theta_1(\eta)}, \frac{\theta_2(2z)}{\theta_2(\eta)}, \frac{\theta_3(2z)}{\theta_3(\eta)}, \frac{\theta_4(2z)}{\theta_4(\eta)}\right) = (Z_0, Z_1, Z_2, Z_3)$$

and $\psi(z) = \varphi(z - \eta/2)$.

Proof Up to an affine transformation, φ (and ψ are) is the classical projective embedding of $\mathbb{C}/L$ in $P_3(\mathbb{C})$. Thus we only need to check that the biquadratic curve $\operatorname{Im}\varphi = \operatorname{Im}\psi$ is given by (3.20). It is thus enough to check (3.20) on $\varphi(z)$. This follows from the basic relations

$$\theta_2^2(0)\theta_3^2(z) = \theta_2^2(z)\theta_3^2(0) + \theta_4^2(0)\theta_1^2(z) \tag{3.22}$$

and

$$\theta_4^2(z)\theta_3^2(0) = \theta_1^2(z)\theta_2^2(0) + \theta_3^2(z)\theta_4^2(0) \tag{3.23}$$

which one uses to check $\frac{Z_0^2 - Z_1^2}{\alpha} = \frac{Z_0^2 - Z_2^2}{\beta}$ and $\frac{Z_0^2 - Z_2^2}{\beta} = \frac{Z_0^2 - Z_3^2}{\gamma}$ respectively.$\square$

The elements of S are obtained from the following values of z

$$\psi(\eta) = p_0,\ \psi(\eta + \frac{1}{2}) = p_1,\ \psi(\eta + \frac{1}{2} + \frac{\omega}{2}) = p_2,\ \psi(\eta + \frac{\omega}{2}) = p_3 \tag{3.24}$$

and

$$\psi(0) = q_0,\ \psi(\frac{1}{2}) = q_1,\ \psi(\frac{1}{2} + \frac{\omega}{2}) = q_2,\ \psi(\frac{\omega}{2}) = q_3. \tag{3.25}$$

(We used $M^{-1}.\psi$ to go back to the coordinates u_μ).
Let $H \sim \mathbb{Z}_2 \times \mathbb{Z}_2$ be the Klein subgroup of the symmetric group $\mathfrak{S}_4$ acting on $P_3(\mathbb{C})$ by permutation of the coordinates (u_0, u_1, u_2, u_3).
For ρ in H one has $\Phi \circ \rho = \Phi$, so that ρ defines for each $\mathbf{u}$ an automorphism of $F_\mathbf{u}$. For ρ in H the matrix $M\rho M^{-1}$ is diagonal with ± 1 on the diagonal and the quasiperiodicity of the θ-functions allows to check that these automorphisms are translations on $F_\mathbf{u}$ by the following 2-torsion elements of $\mathbb{C}/L$,

$$\begin{aligned} \rho &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \end{pmatrix} \quad \text{is translation by } \tfrac{1}{2}, \\ \rho &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \end{pmatrix} \quad \text{is translation by } \tfrac{1}{2} + \tfrac{\omega}{2} \end{aligned} \tag{3.26}$$

Let $\mathcal{O} \subset P_3(\mathbb{C})$ be the complement of the 4 hyperplanes $\{u_\mu = 0\}$ with $\mu \in \{0, 1, 2, 3\}$. Then $(u_0, u_1, u_2, u_3) \mapsto (u_0^{-1}, u_1^{-1}, u_2^{-1}, u_3^{-1})$ defines an involutive automorphism I of $\mathcal{O}$ and since one has

$$(u_0^{-1} + u_k^{-1})(u_\ell^{-1} + u_m^{-1}) = (u_0 u_1 u_2 u_3)^{-1}(u_0 + u_k)(u_\ell + u_m) \tag{3.27}$$

it follows that $\Phi \circ I = \Phi$, so that I defines for each $\mathbf{u} \in \mathcal{O} \backslash \{q_0, q_1, q_2, q_3\}$ an involutive automorphism of $F_\mathbf{u} \cap \mathcal{O}$ which extends canonically to $F_\mathbf{u}$, in fact,

Proposition 4 *The restriction of I to $F_\mathbf{u}$ is the symmetry $\psi(z) \mapsto \psi(-z)$ around any of the points $q_\mu \in F_\mathbf{u}$ in the elliptic curve $F_\mathbf{u}$.*

This symmetry, as well as the above translations by two torsion elements does not refer to a choice of origin in the curve $F_{\mathbf{u}}$. The proof follows from identities on theta functions.

Let $T := \{\mathbf{u} \,|\, |u_\mu| = 1 \ \forall \mu\}$. By section 2 the torus T gives a covering of the real moduli space $\mathcal{M}$. For $\mathbf{u} \in T$, the point $\Phi(\mathbf{u})$ is real with projective coordinates

$$\Phi(\mathbf{u}) = (s_1, s_2, s_3)\,, \quad s_k := 1 + t_\ell\, t_m\,, \quad t_k := \tan(\varphi_k - \varphi_0) \tag{3.28}$$

The corresponding fiber $F_{\mathbf{u}}$ is stable under complex conjugation $\mathbf{v} \mapsto \overline{\mathbf{v}}$ and the intersection of $F_{\mathbf{u}}$ with the real moduli space is given by,

$$F_{\mathbb{T}}(\mathbf{u}) = F_{\mathbf{u}} \cap T = \{\mathbf{v} \in F_{\mathbf{u}} | I(\mathbf{v}) = \overline{\mathbf{v}}\} \tag{3.29}$$

The curve $F_{\mathbf{u}}$ is defined over $\mathbb{R}$ and (3.29) determines its purely <u>imaginary</u> points. Note that $F_{\mathbb{T}}(\mathbf{u})$ (3.29) is invariant under the Klein group H and thus has two connected components, we let $F_{\mathbb{T}}(\mathbf{u})^0$ be the component containing q_0. The real points, $\{\mathbf{v} \in F_{\mathbf{u}} | \mathbf{v} = \overline{\mathbf{v}}\} = F_{\mathbf{u}} \cap P_3(\mathbb{R})$ of $F_{\mathbf{u}}$ do play a complementary role in the characteristic variety (Proposition 12).

4 Generic Fiber = Characteristic Variety

Let us recall the definition of the geometric data $\{E\,, \sigma\,, \mathcal{L}\}$ for quadratic algebras. Let $\mathcal{A} = A(V, R) = T(V)/(R)$ be a quadratic algebra where V is a finite-dimensional complex vector space and where (R) is the two-sided ideal of the tensor algebra $T(V)$ of V generated by the subspace R of $V \otimes V$. Consider the subset of $V^* \times V^*$ of pairs (α, β) such that

$$\langle \omega, \alpha \otimes \beta \rangle = 0, \ \ \alpha \neq 0, \beta \neq 0 \tag{4.1}$$

for any $\omega \in R$. Since R is homogeneous, (4.1) defines a subset

$$\Gamma \subset P(V^*) \times P(V^*)$$

where $P(V^*)$ is the complex projective space of one-dimensional complex subspaces of V^*. Let E_1 and E_2 be the first and the second projection of Γ in $P(V^*)$. It is usually assumed that they coincide i.e. that one has

$$E_1 = E_2 = E \subset P(V^*) \tag{4.2}$$

and that the correspondence σ with graph Γ is an automorphism of E, $\mathcal{L}$ being the pull-back on E of the dual of the tautological line bundle of $P(V^*)$. The algebraic variety E is refered to as the characteristic variety. In many cases E is the union of an elliptic curve with a finite number of points which are invariant by σ. This is the case for $\mathcal{A}_{\mathbf{u}} = C_{\mathrm{alg}}(\mathbb{C}^4(\mathbf{u}))$ at generic $\mathbf{u}$ since it then reduces to the Sklyanin algebra for which this is known [14], [19]. When $\mathbf{u}$ is non generic, e.g. when the isomorphism with the Sklyanin algebra breaks down, the situation is more involved, and the characteristic variety can be as large as $P_3(\mathbb{C})$. This is described in Part I, where we gave a complete description of the geometric datas. Our aim now is to show that for $\mathbf{u} \in P_3(\mathbb{C})$ generic, there is an astute choice of generators of the quadratic algebra $\mathcal{A}_{\mathbf{u}} = C_{\mathrm{alg}}(\mathbb{C}^4(\mathbf{u}))$ for which the characteristic variety $E_{\mathbf{u}}$ actually coincides with the fiber variety $F_{\mathbf{u}}$ and to identify the corresponding automorphism σ. Since this coincidence no longer holds for non-generic values it is a quite remarkable fact which we first noticed by comparing the j-invariants of these two elliptic curves.

Let $\mathbf{u} \in P_3(\mathbb{C})$ be generic, we perform the following change of generators

$$\begin{aligned}
y_0 &= \sqrt{u_1 - u_2}\,\sqrt{u_2 - u_3}\,\sqrt{u_3 - u_1} \quad Y_0 \\
y_1 &= \sqrt{u_0 + u_2}\,\sqrt{u_2 - u_3}\,\sqrt{u_0 + u_3} \quad Y_1 \\
y_2 &= \sqrt{u_0 + u_3}\,\sqrt{u_3 - u_1}\,\sqrt{u_0 + u_1} \quad Y_2 \\
y_3 &= \sqrt{u_0 + u_1}\,\sqrt{u_1 - u_2}\,\sqrt{u_0 + u_2} \quad Y_3
\end{aligned} \tag{4.3}$$

We let $J_{\ell m}$ be as before, given by (3.9)

$$J_{12} = \frac{\alpha - \beta}{\gamma}, \quad J_{23} = \frac{\beta - \gamma}{\alpha}, \quad J_{31} = \frac{\gamma - \alpha}{\beta} \tag{4.4}$$

with α, β, γ given by (3.11). Finally let e_ν be the 4 points of $P_3(\mathbb{C})$ whose homogeneous coordinates (Z_μ) all vanish but one.

Theorem 5 *1) In terms of the Y_μ, the relations of $\mathcal{A}_{\mathbf{u}}$ take the form*

$$[Y_0, Y_k]_- = [Y_\ell, Y_m]_+ \tag{4.5}$$

$$[Y_\ell, Y_m]_- = -J_{\ell m}[Y_0, Y_k]_+ \tag{4.6}$$

for any $k \in \{1, 2, 3\}$, (k, ℓ, m) being a cyclic permutation of (1,2,3)
2) The characteristic variety $E_{\mathbf{u}}$ is the union of $F_{\mathbf{u}}$ with the 4 points e_ν.
3) The automorphism σ of the characteristic variety $E_{\mathbf{u}}$ is given by

$$\psi(z) \mapsto \psi(z - \eta) \tag{4.7}$$

on $F_{\mathbf{u}}$ and $\sigma = Id$ on the 4 points e_ν.
4) The automorphism σ is the restriction to $F_{\mathbf{u}}$ of a birational automorphism of $P_3(\mathbb{C})$ independent of $\mathbf{u}$ and defined over $\mathbb{Q}$.

The resemblance between the above presentation and the Sklyanin one (3.4), (3.5) is misleading, for the latter all the characteristic varieties are contained in the same quadric (cf. [19] §2.4)

$$\sum x_\mu^2 = 0$$

and cant of course form a net of essentially disjoint curves.

Proof By construction $E_{\mathbf{u}} = \{Z \mid \text{Rank } N(Z) < 4\}$ where

$$N(Z) = \begin{pmatrix} Z_1 & -Z_0 & Z_3 & Z_2 \\ Z_2 & Z_3 & -Z_0 & Z_1 \\ Z_3 & Z_2 & Z_1 & -Z_0 \\ (\beta-\gamma)Z_1 & (\beta-\gamma)Z_0 & -\alpha Z_3 & \alpha Z_2 \\ (\gamma-\alpha)Z_2 & \beta Z_3 & (\gamma-\alpha)Z_0 & -\beta Z_1 \\ (\alpha-\beta)Z_3 & -\gamma Z_2 & \gamma Z_1 & (\alpha-\beta)Z_0 \end{pmatrix} \tag{4.8}$$

One checks that it is the union of the fiber $F_{\mathbf{u}}$ (in the generic case) with the above 4 points. The automorphism σ of the characteristic variety $E_{\mathbf{u}}$ is given by definition by the equation,

$$N(Z)\,\sigma(Z) = 0 \tag{4.9}$$

where $\sigma(Z)$ is the column vector $\sigma(Z_\mu) := M \cdot \sigma(\mathbf{u})$ (in the variables Z_λ). One checks that $\sigma(Z)$ is already determined by the equations in (4.9) corresponding to the first three lines in $N(Z)$ which are independent of α, β, γ (see below). Thus σ is in fact an automorphism of $P_3(\mathbb{C})$ which is the identity on the above four points and which restricts as automorphism of $F_{\mathbf{u}}$ for each

$\mathbf{u}$ generic. One checks that σ is the product of two involutions which both restrict to $E_{\mathbf{u}}$ (for $\mathbf{u}$ generic)

$$\sigma = I \circ I_0 \tag{4.10}$$

where I is the involution of the end of section 3 corresponding to $u_\mu \mapsto u_\mu^{-1}$ and where I_0 is given by

$$I_0(Z_0) = -Z_0,\ \ I_0(Z_k) = Z_k \tag{4.11}$$

for $k \in \{1,2,3\}$ and which restricts obviously to $E_{\mathbf{u}}$ in view of (3.17). Both I and I_0 are the identity on the above four points and since I_0 induces the symmetry $\varphi(z) \mapsto \varphi(-z)$ around $\varphi(0) = \psi(\eta/2)$ (proposition 3) one gets the result using proposition 4.

The fact that σ does not depend on the parameters α, β, γ plays an important role. Explicitly we get from the first 3 equations (4.9)

$$\sigma(Z)_\mu = \eta_{\mu\mu}(Z_\mu^3 - Z_\mu \sum_{\nu \neq \mu} Z_\nu^2 - 2 \prod_{\lambda \neq \mu} Z_\lambda) \tag{4.12}$$

for $\mu \in \{0,1,2,3\}$, where $\eta_{00} = 1$ and $\eta_{nn} = -1$ for $n \in \{1,2,3\}$. $\square$

5 Central Quadratic Forms and Generalised Cross-Products

Let $\mathcal{A} = A(V,R) = T(V)/(R)$ be a quadratic algebra. Its geometric data $\{E\,,\,\sigma\,,\,\mathcal{L}\}$ is defined in such a way that $\mathcal{A}$ maps homomorphically to a cross-product algebra obtained from sections of powers of the line bundle $\mathcal{L}$ on powers of the correspondence σ ([3]).

We shall begin by a purely algebraic result which considerably refines the above homomorphism and lands in a richer cross-product. We use the notations of section 4 for general quadratic algebras.

Definition 6 *Let $Q \in S^2(V)$ be a symmetric bilinear form on V^* and C a component of $E \times E$. We shall say that Q is* <u>*central*</u> *on C iff for all (Z, Z') in C and $\omega \in R$ one has,*

$$\omega(Z, Z')\, Q(\sigma(Z'), \sigma^{-1}(Z)) + Q(Z, Z')\, \omega(\sigma(Z'), \sigma^{-1}(Z)) = 0$$

Let C be a component of $E \times E$ globally invariant under the map

$$\tilde{\sigma}(Z, Z') := (\sigma(Z), \sigma^{-1}(Z')) \tag{5.1}$$

Given a quadratic form Q central and not identically zero on the component C, we define as follows an algebra C_Q as a generalised cross-product of the ring of meromorphic functions on C by the transformation $\tilde{\sigma}$. Let L, $L' \in V$ be such that $L(Z)\, L'(Z')$ does not vanish identically on C. We adjoin two generators W_L and $W'_{L'}$ which besides the usual cross-product rules,

$$W_L\, f = (f \circ \tilde{\sigma})\, W_L\,, \quad W'_{L'}\, f = (f \circ \tilde{\sigma}^{-1})\, W'_{L'}\,, \quad \forall f \in C \tag{5.2}$$

fulfill the following relations,

$$W_L\, W'_{L'} := \pi(Z, Z')\,, \qquad W'_{L'}\, W_L := \pi(\sigma^{-1}(Z), \sigma(Z')) \tag{5.3}$$

where the function $\pi(Z, Z')$ is given by the ratio,

$$\pi(Z, Z') := \frac{L(Z)\, L'(Z')}{Q(Z, Z')} \tag{5.4}$$

The a priori dependence on L, L' is eliminated by the rules,

$$W_{L_2} := \frac{L_2(Z)}{L_1(Z)}\, W_{L_1} \qquad W'_{L'_2} := W'_{L'_1}\, \frac{L'_2(Z')}{L'_1(Z')} \tag{5.5}$$

which allow to adjoin all W_L and $W'_{L'}$ for L and L' not identically zero on the projections of C, without changing the algebra and provides an intrinsic definition of C_Q.

Our first result is

Lemma 7 *Let Q be central and not identically zero on the component C.*
(i) The following equality defines a homomorphism ρ: $\mathcal{A} \mapsto C_Q$

$$\sqrt{2}\,\rho(Y) := \frac{Y(Z)}{L(Z)}\,W_L + W'_{L'}\,\frac{Y(Z')}{L'(Z')}\,, \qquad \forall Y \in V$$

(ii) If $\sigma^4 \neq \mathbb{1}$, then $\rho(Q) = 1$ where Q is viewed as an element of $T(V)/(R)$.

Formula (i) is independent of L, L' using (5.5) and reduces to $W_Y + W'_Y$ when Y is non-trivial on the two projections of C. It is enough to check that the $\rho(Y) \in C_Q$ fulfill the quadratic relations $\omega \in R$. We view $\omega \in R$ as a bilinear form on V^*. The vanishing of the terms in W^2 and in W'^2 is automatic by construction of the characteristic variety. The vanishing of the sum of terms in $W\,W', W'\,W$ follows from definition 6.
Let $\mathcal{A}_{\mathbf{u}} = C_{\mathrm{alg}}(\mathbb{C}^4(\mathbf{u}))$ at generic $\mathbf{u}$, then the center of $\mathcal{A}_{\mathbf{u}}$ is generated by the three linearly dependent quadratic elements

$$Q_m \;:=\; J_{k\ell}\,(Y_0^2 \,+\, Y_m^2) \,+\, Y_k^2 \,-\, Y_\ell^2 \tag{5.6}$$

with the notations of theorem 5.

Proposition 8 *Let $\mathcal{A}_{\mathbf{u}} = C_{\mathrm{alg}}(\mathbb{C}^4(\mathbf{u}))$ at generic $\mathbf{u}$, then each Q_m is central on $F_{\mathbf{u}} \times F_{\mathbf{u}}$ ($\subset E_{\mathbf{u}} \times E_{\mathbf{u}}$).*

One uses (4.10) to check the algebraic identity. Together with lemma 7 this yields non trivial homomorphisms of $\mathcal{A}_{\mathbf{u}}$ whose unitarity will be analysed in the next section. Note that for a general quadratic algebra $\mathcal{A} = A(V, R) = T(V)/(R)$ and a quadratic form $Q \in S^2(V)$, such that $Q \in \mathrm{Center}(\mathcal{A})$, it does not automatically follow that Q is central on $E \times E$. For instance Proposition 8 no longer holds on $F_{\mathbf{u}} \times \{e_\nu\}$ where e_ν is any of the four points of $E_{\mathbf{u}}$ not in $F_{\mathbf{u}}$.

6 Positive Central Quadratic Forms on Quadratic $*$-Algebras

The algebra $\mathcal{A}_{\mathbf{u}}$, $\mathbf{u} \in T$ is by construction a *quadratic* $*$*-algebra* i.e. a quadratic complex algebra $\mathcal{A} = A(V, R)$ which is also a $*$-algebra with involution $x \mapsto x^*$ preserving the subspace V of generators. Equivalently one can take the generators of $\mathcal{A}$ (spanning V) to be hermitian elements of $\mathcal{A}$. In such a case the complex finite-dimensional vector space V has a real structure given by the antilinear involution $v \mapsto j(v)$ obtained by restriction of $x \mapsto x^*$. Since one has $(xy)^* = y^* x^*$ for $x, y \in \mathcal{A}$, it follows that the set R of relations satisfies

$$(j \otimes j)(R) = t(R) \tag{6.1}$$

in $V \otimes V$ where $t : V \otimes V \to V \otimes V$ is the transposition $v \otimes w \mapsto t(v \otimes w) = w \otimes v$. This implies

Lemma 9 *The characteristic variety is stable under the involution $Z \mapsto j(Z)$ and one has*

$$\sigma(j(Z)) = j(\sigma^{-1}(Z))$$

We let C be as above an invariant component of $E \times E$ we say that C is j-real when it is globally invariant under the involution

$$\tilde{j}(Z, Z') := (j(Z'), j(Z)) \tag{6.2}$$

By lemma 9 this involution commutes with the automorphism $\tilde{\sigma}$ (5.1) and the following turns the cross-product C_Q into a $*$-algebra,

$$f^*(Z, Z') := \overline{f(\tilde{j}(Z, Z'))}\,, \qquad (W_L)^* = W'_{j(L)}\,, \qquad (W'_{L'})^* = W_{j(L')} \tag{6.3}$$

provided that $Q \in S^2(V)$ fulfills $Q = Q^*$. We use the transpose of j, so that

$$j(L)(Z) = \overline{L(j(Z))}\,, \qquad \forall Z \in V^*. \tag{6.4}$$

The homomorphism ρ of lemma 7 is a $*$-homomorphism. Composing ρ with the restriction to the subset $K = \{(Z, Z') \in C \,|\, Z' = j(Z)\}$ one obtains a $*$-homomorphism θ of $\mathcal{A} = A(V, R)$ to a twisted cross-product C^*-algebra, $C(K) \times_{\sigma, \mathcal{L}} \mathbb{Z}$. Given a compact space K, an homeomorphism σ of K and a hermitian line bundle $\mathcal{L}$ on K we define the C^*-algebra $C(K) \times_{\sigma, \mathcal{L}} \mathbb{Z}$ as the twisted cross-product of $C(K)$ by the Hilbert C^*-bimodule associated to $\mathcal{L}$ and σ ([1], [15]). We let for each $n \geq 0$, $\mathcal{L}^{\sigma^n}$ be the hermitian line bundle pullback of $\mathcal{L}$ by σ^n and (cf. [3], [19])

$$\mathcal{L}_n := \mathcal{L} \otimes \mathcal{L}^{\sigma} \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}} \tag{6.5}$$

We first define a $*$-algebra as the linear span of the monomials

$$\xi\, W^n\,, \quad W^{*n}\, \eta^*\,, \quad \xi\,, \eta \in C(K, \mathcal{L}_n) \tag{6.6}$$

with product given as in ([3], [19]) for $(\xi_1\, W^{n_1})\, (\xi_2\, W^{n_2})$ so that

$$(\xi_1\, W^{n_1})\, (\xi_2\, W^{n_2}) := (\xi_1 \otimes (\xi_2 \circ \sigma^{n_1}))\, W^{n_1+n_2} \tag{6.7}$$

We use the hermitian structure of $\mathcal{L}_n$ to give meaning to the products $\eta^*\, \xi$ and $\xi\ \eta^*$ for $\xi\,, \eta \in C(K, \mathcal{L}_n)$. The product then extends uniquely to an associative product of $*$-algebra fulfilling the following additional rules

$$(W^{*k}\, \eta^*)\, (\xi\, W^k) := \ (\eta^*\, \xi) \circ \sigma^{-k}\,, \qquad (\xi\, W^k)\, (W^{*k}\, \eta^*) \ := \ \xi\ \eta^* \tag{6.8}$$

The C^*-norm of $C(K) \times_{\sigma, \mathcal{L}} \mathbb{Z}$ is defined as for ordinary cross-products and due to the amenability of the group $\mathbb{Z}$ there is no distinction between the reduced and maximal norms. The latter is obtained as the supremum of the norms in involutive representations in Hilbert space. The natural positive conditional expectation on the subalgebra $C(K)$ shows that the C^*-norm restricts to the usual sup norm on $C(K)$.

To lighten notations in the next statement we abreviate $j(Z)$ as $\bar{Z}$,

Theorem 10 *Let $K \subset E$ be a compact σ-invariant subset and Q be central and strictly positive on $\{(Z, \bar{Z});\, Z \in K\}$. Let $\mathcal{L}$ be the restriction to K of the dual of the tautological line bundle on $P(V^*)$ endowed with the unique hermitian metric such that*

$$\langle L, L' \rangle = \frac{L(Z)\,\overline{L'(Z)}}{Q(Z,\, \bar{Z})} \qquad L, L' \in V, \quad Z \in K$$

(i) The equality $\sqrt{2}\,\theta(Y) := Y\,W + W^*\,\bar{Y}^*$ yields a $*$-homomorphism

$$\theta : \mathcal{A} = A(V, R) \mapsto C(K) \times_{\sigma,\, \mathcal{L}} \mathbb{Z}$$

(ii) For any $Y \in V$ the C^*-norm of $\theta(Y)$ fulfills

$$\mathrm{Sup}_K \|Y\| \leq \sqrt{2} \|\, \theta(Y) \| \leq 2\, \mathrm{Sup}_K \|Y\|$$

(iii) If $\sigma^4 \neq \mathbb{1}$, then $\theta(Q) = 1$ where Q is viewed as an element of $T(V)/(R)$.

We shall now apply this general result to the algebras $\mathcal{A}_{\mathbf{u}}$, $\mathbf{u} \in T$. We take the quadratic form

$$Q(X,\, X') := \sum X_\mu\, X'_\mu \tag{6.9}$$

in the x-coordinates, so that Q is the canonical central element defining the sphere $S^3_{\mathbf{u}}$ by the equation $Q = 1$. Proposition 8 shows that Q is central on $F_{\mathbf{u}} \times F_{\mathbf{u}}$ for generic $\mathbf{u}$. The positivity of Q is automatic since in the x-coordinates the involution $j_{\mathbf{u}}$ coming from the involution of the quadratic $*$-algebra $\mathcal{A}_{\mathbf{u}}$ is simply complex conjugation $j_{\mathbf{u}}(Z) = \bar{Z}$, so that $Q(X,\, j_{\mathbf{u}}(X)) > 0$ for $X \neq 0$. We thus get,

Corollary 11 *Let $K \subset F_{\mathbf{u}}$ be a compact σ-invariant subset. The homomorphism θ of Theorem 10 is a unital $*$-homomorphism from $C_{\mathrm{alg}}(S^3_{\mathbf{u}})$ to the cross-product $C^\infty(K) \times_{\sigma,\, \mathcal{L}} \mathbb{Z}$.*

This applies in particular to $K = F_{\mathbf{u}}$. It follows that one obtains a non-trivial C^*-algebra $C^*(S^3_{\mathbf{u}})$ as the completion of $C_{\rm alg}(S^3_{\mathbf{u}})$ for the semi-norm,

$$\|P\| := \operatorname{Sup}\|\pi(P)\| \tag{6.10}$$

where π varies through all unitary representations of $C_{\rm alg}(S^3_{\mathbf{u}})$.
To analyse the compact σ-invariant subsets of $F_{\mathbf{u}}$ for generic $\mathbf{u}$, we distinguish two cases. First note that the real curve $F_{\mathbf{u}} \cap P_3(\mathbb{R})$ is non empty (it contains p_0), and has two connected components since it is invariant under the Klein group H (3.26).

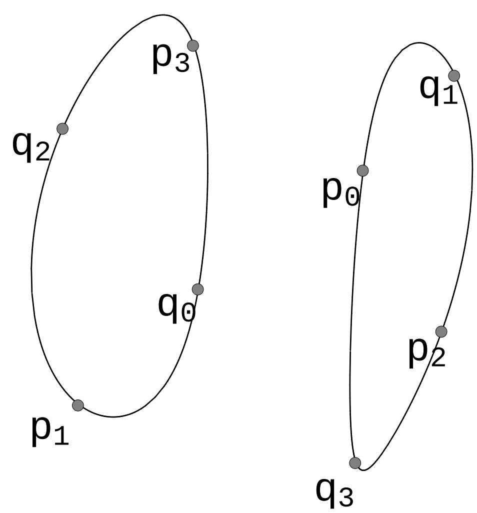

Figure 2: The Elliptic Curve $F_{\mathbf{u}} \cap P_3(\mathbb{R})$ (odd case)

We say that $\mathbf{u} \in T$ is <u>even</u> when σ preserves each of the two connected components of the real curve $F_{\mathbf{u}} \cap P_3(\mathbb{R})$ and <u>odd</u> when it permutes them (cf. Figure 2). A generic $\mathbf{u} \in T$ is even (cf. Figure 1) iff the s_k of (3.28)

($s_k = 1 + t_\ell\, t_m$, $t_k = \tan \varphi_k$) have the same sign. In that case σ is the square of a real translation κ of the elliptic curve $F_{\mathbf{u}}$ preserving $F_{\mathbf{u}} \cap P_3(\mathbb{R})$.

Proposition 12 *Let $\mathbf{u} \in T$ be generic and even.*
(i) Each connected component of $F_{\mathbf{u}} \cap P_3(\mathbb{R})$ is a minimal compact σ-invariant subset.
(ii) Let $K \subset F_{\mathbf{u}}$ be a compact σ-invariant subset, then K is the sum in the elliptic curve $F_{\mathbf{u}}$ with origin p_0 of $K_{\mathbb{T}} = K \cap F_{\mathbb{T}}(\mathbf{u})^0$ (cf. 3.29) with the component $C_{\mathbf{u}}$ of $F_{\mathbf{u}} \cap P_3(\mathbb{R})$ containing p_0.
(iii) The cross-product $C(F_{\mathbf{u}}) \times_{\sigma, \mathcal{L}} \mathbb{Z}$ is isomorphic to the mapping torus of the automorphism β of the noncommutative torus $\mathbb{T}^2_\eta = C_{\mathbf{u}} \times_\sigma \mathbb{Z}$ acting on the generators by the matrix $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$.

More precisely with U_j the generators one has

$$\beta(U_1) := U_1 \,, \qquad \beta(U_2) := U_1^4\, U_2 \tag{6.11}$$

The mapping torus of the automorphism β is given by the algebra of continuous maps $s \in \mathbb{R} \mapsto x(s) \in C(\mathbb{T}^2_\eta)$ such that $x(s+1) = \beta(x(s))$, $\forall s \in \mathbb{R}$.

Corollary 13 *Let $\mathbf{u} \in T$ be generic and even, then $F_{\mathbf{u}} \times_{\sigma, \mathcal{L}} \mathbb{Z}$ is a noncommutative 3-manifold with an elliptic action of the three dimensional Heisenberg Lie algebra $\mathfrak{h}_3$ and an invariant trace τ.*

We refer to [16] and [2] where these noncommutative manifolds were introduced and analysed in terms of crossed products by Hilbert C^*-bimodules. One can construct directly the action of $\mathfrak{h}_3$ on $C^\infty(F_{\mathbf{u}}) \times_{\sigma, \mathcal{L}} \mathbb{Z}$ by choosing a constant (translation invariant) curvature connection ∇, compatible with the metric, on the hermitian line bundle $\mathcal{L}$ on $F_{\mathbf{u}}$ (viewed in the C^∞-category not in the holomorphic one). The two covariant differentials ∇_j corresponding to

the two vector fields X_j on $F_{\mathbf{u}}$ generating the translations of the elliptic curve, give a natural extension of X_j as the unique derivations δ_j of $C^\infty(F_{\mathbf{u}}) \times_{\sigma,\mathcal{L}} \mathbb{Z}$ fulfilling the rules,

$$\begin{aligned} \delta_j(f) &= X_j(f)\,, \quad \forall f \in C^\infty(F_{\mathbf{u}}) \\ \delta_j(\xi\, W) &= \nabla_j(\xi)\, W\,, \quad \forall \xi \in C^\infty(F_{\mathbf{u}}\,, \mathcal{L}) \end{aligned} \tag{6.12}$$

We let δ be the unique derivation of $C^\infty(F_{\mathbf{u}}) \times_{\sigma,\mathcal{L}} \mathbb{Z}$ corresponding to the grading by powers of W. It vanishes on $C^\infty(F_{\mathbf{u}})$ and fulfills

$$\delta(\xi\, W^k) = i\, k\, \xi\, W^k \qquad \delta(W^{*k}\, \eta^*) = -i\, k\, W^{*k}\, \eta^* \tag{6.13}$$

Let $i\,\kappa$ be the constant curvature of the connection ∇, one gets

$$[\delta_1,\, \delta_2] = \kappa\, \delta\,, \quad [\delta,\, \delta_j] = 0 \tag{6.14}$$

which provides the required action of the Lie algebra $\mathfrak{h}_3$ on $C^\infty(F_{\mathbf{u}}) \times_{\sigma,\mathcal{L}} \mathbb{Z}$. Integration on the translation invariant volume form dv of $F_{\mathbf{u}}$ gives the $\mathfrak{h}_3$-invariant trace τ,

$$\begin{aligned} \tau(f) &= \int f dv\,, \quad \forall f \in C^\infty(F_{\mathbf{u}}) \\ \tau(\xi\, W^k) &= \tau(W^{*k}\, \eta^*) = 0\,, \quad \forall k \neq 0 \end{aligned} \tag{6.15}$$

It follows in particular that the results of [4] apply to obtain the calculus. In particular the following gives the "fundamental class" as a 3-cyclic cocycle,

$$\tau_3(a_0,\, a_1,\, a_2,\, a_3) = \sum \epsilon_{ijk}\, \tau(a_0\, \delta_i(a_1)\, \delta_j(a_2)\, \delta_k(a_3)) \tag{6.16}$$

where the δ_j are the above derivations with $\delta_3 := \delta$.

We shall in fact describe the same calculus in greater generality in the last section which will be devoted to the computation of the Jacobian of the homomorphism θ of corollary 11.

Similar results hold in the odd case. Then $F_{\mathbf{u}} \cap P_3(\mathbb{R})$ is a minimal compact σ-invariant subset, any compact σ-invariant subset $K \subset F_{\mathbf{u}}$ is the sum in the elliptic curve $F_{\mathbf{u}}$ with origin p_0 of $F_{\mathbf{u}} \cap P_3(\mathbb{R})$ with $K_{\mathbb{T}} = K \cap F_{\mathbb{T}}(\mathbf{u})^0$ but the latter is automatically invariant under the subgroup $H_0 \subset H$ of order 2 of the Klein group H (3.26)

$$H_0 := \{h \in H \,|\, h(F_{\mathbb{T}}(\mathbf{u})^0) = F_{\mathbb{T}}(\mathbf{u})^0\} \tag{6.17}$$

The group law in $F_{\mathbf{u}}$ is described geometrically as follows. It involves the point q_0. The sum $z = x+y$ of two points x and y of $F_{\mathbf{u}}$ is $z = I_0(w)$ where w is the 4th point of intersection of $F_{\mathbf{u}}$ with the plane determined by the three points $\{q_0, x, y\}$. It commutes by construction with complex conjugation so that $\overline{x+y} = \overline{x} + \bar{y}\,, \quad \forall x,\, y \in F_{\mathbf{u}}$.

By lemma 9 the translation σ is imaginary for the canonical involution $j_{\mathbf{u}}$. In terms of the coordinates Z_μ this involution is described as follows, using (4.3) (multiplied by $e^{i(\pi/4-\varphi_1-\varphi_2-\varphi_3)}2^{-3/2}$) to change variables. Among the 3 real numbers

$$\lambda_k \quad = \quad \cos\varphi_\ell \, \cos\varphi_m \, \sin(\varphi_\ell - \varphi_m)\,, \qquad k \in \{1,2,3\}$$

two have the same sign ϵ and one, λ_k, $k \in \{1,2,3\}$, the opposite sign. Then

$$j_{\mathbf{u}} = \epsilon\, I_k \,\circ\, c \tag{6.18}$$

where c is complex conjugation on the real elliptic curve $F_{\mathbf{u}}$ (section 3) and I_μ the involution

$$I_\mu(Z_\mu) = -Z_\mu,\; I_\mu(Z_\nu) = Z_\nu\,, \quad \nu \neq \mu \tag{6.19}$$

The index k and the sign ϵ remain constant when $\mathbf{u}$ varies in each of the four components of the complement of the four points q_μ in $F_{\mathbb{T}}(\mathbf{u})$. The sign ϵ

matters for the action of $j_{\mathbf{u}}$ on linear forms as in (6.4), but is irrelevant for the action on $F_{\mathbf{u}}$. Each involution I_μ is a symmetry $z \mapsto p - z$ in the elliptic curve $F_{\mathbf{u}}$ and the products $I_\mu \circ I_\nu$ form the Klein subgroup H (3.26) acting by translations of order two on $F_{\mathbf{u}}$.

The quadratic form Q of (6.9) is given in the new coordinates by,

$$Q = \left(\prod \cos^2 \varphi_\ell\right) \sum t_k\, s_k\, Q_k \tag{6.20}$$

with $s_k := 1 + t_\ell\, t_m$, $t_k := \tan \varphi_k$ and Q_k defined by (5.6).

Let $\mathbf{u} \in T$ be generic and even and $v \in F_{\mathbb{T}}(\mathbf{u})^0$. Let $K(v) = v + C_{\mathbf{u}}$ be the minimal compact σ-invariant subset containing v (Proposition 12 (ii)). By Corollary 11 we get a homomorphism,

$$\theta_v \;:\; C_{\text{alg}}(S^3_{\mathbf{u}}) \mapsto C^\infty(\mathbb{T}^2_\eta) \tag{6.21}$$

whose non-triviality will be proved below in corollary 17. We shall first show (Theorem 14) that it transits through the cross-product of the field K_q of meromorphic functions on the elliptic curve by the subgroup of its Galois group $\mathrm{Aut}_{\mathbb{C}}(K_q)$ generated by the translation σ.

For $Z = v + z$, $z \in C_{\mathbf{u}}$, one has using (6.18) and (3.29),

$$j_{\mathbf{u}}(Z) = I_\mu(Z - v) - I(v) \tag{6.22}$$

which is a rational function $r(v, Z)$. Fixing $\mathbf{u}$, v and substituting Z and $Z' = r(v, Z)$ in the formulas (5.3) and (5.4) of lemma 7 with L real such that $0 \notin L(K(v))$, $L' = \epsilon\, L \circ I_\mu$ and Q given by (6.20) we obtain rational formulas for a homomorphism $\tilde{\theta_v}$ of $C_{\text{alg}}(S^3_{\mathbf{u}})$ to the generalised cross-product of the field K_q of meromorphic functions $f(Z)$ on the elliptic curve $F_{\mathbf{u}}$ by σ. The generalised cross-product rule (5.3) is given by $W\, W' := \gamma(Z)$ where γ is a rational function. Similarly $W'\, W := \gamma(\sigma^{-1}(Z))$. Using integration on the cycle $K(v)$ to obtain a trace, together with corollary 11, we get,

Theorem 14 *The homomorphism $\theta_v : C_{\mathrm{alg}}(S^3_{\mathbf{u}}) \mapsto C^\infty(\mathbb{T}^2_\eta)$ factorises with a homomorphism $\tilde{\theta}_v : C_{\mathrm{alg}}(S^3_{\mathbf{u}}) \mapsto K_q \times_\sigma \mathbb{Z}$ to the generalised cross-product of the field K_q of meromorphic functions on the elliptic curve $F_{\mathbf{u}}$ by the subgroup of the Galois group $Aut_{\mathbb{C}}(K_q)$ generated by σ. Its image generates the hyperfinite factor of type II_1 after weak closure relative to the trace given by integration on the cycle $K(v)$.*

Elements of K_q with poles on $K(v)$ are unbounded and give elements of the regular ring of affiliated operators, but all elements of $\theta_v(C_{\mathrm{alg}}(S^3_{\mathbf{u}}))$ are regular on $K(v)$. The above generalisation of the cross-product rules (5.3) with the rational formula for $W\,W' := \gamma(Z)$ is similar to the introduction of 2-cocycles in the standard Brauer theory of central simple algebras.

7 The Jacobian of the Covering of $S^3_{\mathbf{u}}$

In this section we shall analyse the morphism of $*$-algebras

$$\theta \,:\, C_{\mathrm{alg}}(S^3_{\mathbf{u}}) \mapsto C^\infty(F_{\mathbf{u}} \times_{\sigma,\mathcal{L}} \mathbb{Z}) \tag{7.1}$$

of Corollary 11, by computing its Jacobian in the sense of noncommutative differential geometry ([5]).

The usual Jacobian of a smooth map $\varphi : M \mapsto N$ of manifolds is obtained as the ratio $\varphi^*(\omega_N)/\omega_M$ of the pullback of the volume form ω_N of the target manifold N with the volume form ω_M of the source manifold M. In noncommutative geometry, differential forms ω of degree d become Hochschild classes $\tilde{\omega} \in HH_d(\mathcal{A})$, $\mathcal{A} = C^\infty(M)$. Moreover since one works with the dual formulation in terms of algebras, the pullback $\varphi^*(\omega_N)$ is replaced by the pushforward $\varphi^t_*(\tilde{\omega}_N)$ under the corresponding transposed morphism of algebras $\varphi^t(f) := f \circ \varphi\,, \quad \forall f \in C^\infty(N)$.

The noncommutative sphere $S^3_{\mathbf{u}}$ admits a canonical "volume form" given by the Hochschild 3-cycle $\mathrm{ch}_{\frac{3}{2}}(U)$. Our goal is to compute the push-forward,

$$\theta_*(\mathrm{ch}_{\frac{3}{2}}(U)) \in HH_3(C^\infty(F_{\mathbf{u}} \times_{\sigma,\mathcal{L}} \mathbb{Z})) \tag{7.2}$$

Let $\mathbf{u}$ be generic and even. The noncommutative manifold $F_{\mathbf{u}} \times_{\sigma,\mathcal{L}} \mathbb{Z}$ is, by Corollary 13, a noncommutative 3-dimensional nilmanifold isomorphic to the mapping torus of an automorphism of the noncommutative 2-torus T^2_η. Its Hohschild homology is easily computed using the corresponding result for the noncommutative torus ([5]). It admits in particular a canonical volume form $V \in HH_3(C^\infty(F_{\mathbf{u}} \times_{\sigma,\mathcal{L}} \mathbb{Z}))$ which corresponds to the natural class in $HH_2(C^\infty(T^2_\eta))$ ([5]). The volume form V is obtained in the cross-product $F_{\mathbf{u}} \times_{\sigma,\mathcal{L}} \mathbb{Z}$ from the translation invariant 2-form dv on $F_{\mathbf{u}}$.
To compare $\theta_*(\mathrm{ch}_{\frac{3}{2}}(U))$ with V we shall pair it with the 3-dimensional Hochschild cocycle $\tau_h \in HH^3(C^\infty(F_{\mathbf{u}} \times_{\sigma,\mathcal{L}} \mathbb{Z}))$ given, for any element h of the center of $C^\infty(F_{\mathbf{u}} \times_{\sigma,\mathcal{L}} \mathbb{Z})$, by

$$\tau_h(a_0, a_1, a_2, a_3) = \tau_3(h\, a_0, a_1, a_2, a_3) \tag{7.3}$$

where $\tau_3 \in HC^3(C^\infty(F_{\mathbf{u}} \times_{\sigma,\mathcal{L}} \mathbb{Z}))$ is the fundamental class in cyclic cohomology defined by (6.16).
By [9], (2.13) p.549, the component $\mathrm{ch}_{\frac{3}{2}}(U)$ of the Chern character is given by,

$$\begin{aligned}\mathrm{ch}_{\frac{3}{2}}(U) = \;& - \sum \epsilon_{\alpha\beta\gamma\delta} \cos(\varphi_\alpha - \varphi_\beta + \varphi_\gamma - \varphi_\delta)\; x^\alpha\, dx^\beta\, dx^\gamma\, dx^\delta \; + \\ & i \sum \sin 2(\varphi_\mu - \varphi_\nu)\; x^\mu\, dx^\nu\, dx^\mu\, dx^\nu \end{aligned} \tag{7.4}$$

where $\varphi_0 := 0$. In terms of the Y_μ one gets,

$$\begin{aligned}\mathrm{ch}_{\frac{3}{2}}(U) \;=\;& \lambda \sum \delta_{\alpha\beta\gamma\delta}\, (s_\alpha - s_\beta + s_\gamma - s_\delta)\; Y_\alpha\, dY_\beta\, dY_\gamma\, dY_\delta \; + \\ & \lambda \sum \epsilon_{\alpha\beta\gamma\delta}\, (s_\alpha - s_\beta)\;\; Y_\gamma\, dY_\delta\, dY_\gamma\, dY_\delta \end{aligned} \tag{7.5}$$

where $s_0 := 0$, $s_k := 1 + t_\ell \, t_m$, $t_k := \tan \varphi_k$ and

$$\delta_{\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta\gamma\delta} \, (n_\alpha - n_\beta + n_\gamma - n_\delta) \tag{7.6}$$

with $n_0 = 0$ and $n_k = 1$. The normalization factor is

$$\lambda = \, -i \, \prod \cos^2(\varphi_k) \, \sin(\varphi_\ell - \varphi_m) \tag{7.7}$$

Formula (7.5) shows that, up to normalization, $\mathrm{ch}_{\frac{3}{2}}(U)$ only depends on the fiber $F_{\mathbf{u}}$ of $\mathbf{u}$.

Let $\mathbf{u} \in T$ be generic, we assume for simplicity that $\mathbf{u}$ is even, there is a similar formula in the odd case. In our case the involutions I and I_0 are conjugate by a real translation κ of the elliptic curve $F_{\mathbf{u}}$ and we let $F_{\mathbf{u}}(0)$ be one of the two connected components of,

$$\{Z \in F_{\mathbf{u}} \,|\, I_0(Z) = \bar{Z}\} \tag{7.8}$$

By Proposition 12 we can identify the center of $C^\infty(F_{\mathbf{u}} \times_{\sigma,\mathcal{L}} \mathbb{Z})$ with $C^\infty(F_{\mathbf{u}}(0))$. We assume for simplicity that $\varphi_j \in [0, \frac{\pi}{2}]$ are in cyclic order $\varphi_k < \varphi_l < \varphi_m$ for some $k \in \{1, 2, 3\}$.

Theorem 15 *Let $h \in$ Center $(C^\infty(F_{\mathbf{u}} \times_{\sigma,\mathcal{L}} \mathbb{Z})) \sim C^\infty(F_{\mathbf{u}}(0))$. Then*

$$\langle ch_{\frac{3}{2}}(U), \tau_h \rangle = \, 6\,\pi\,\Omega \int_{F_{\mathbf{u}}(0)} h(Z)\, dR(Z)$$

where Ω is the period given by the elliptic integral of the first kind,

$$\Omega = \int_{C_{\mathbf{u}}} \frac{Z_0 dZ_k - Z_k dZ_0}{Z_\ell Z_m}$$

and R the rational fraction,

$$R(Z) = \, t_k \, \frac{Z_m^2}{Z_m^2 + \, c_k \, Z_l^2}$$

with $c_k = \mathrm{tg}(\varphi_l) \, \cot(\varphi_k - \varphi_\ell)$.

We first obtained this result by direct computation using the explicit formula (6.16) and the natural constant curvature connection on $\mathcal{L}$ given by the parametrisation of Proposition 3 in terms of θ-functions. The pairing was first expressed in terms of elliptic functions and modular forms, and the conceptual understanding of its simplicity is at the origin of many of the notions developpped in the present paper and in particular of the "rational" formulation of the calculus which will be obtained in the last section. The geometric meaning of Theorem 15 is the computation of the Jacobian in the sense of noncommutative geometry of the morphism θ as explained above. The differential form

$$\omega := \frac{Z_0 dZ_k - Z_k dZ_0}{s_k\, Z_\ell Z_m} \tag{7.9}$$

is independent of k and is, up to scale, the only holomorphic form of type $(1,0)$ on $F_{\mathbf{u}}$, it is invariant under the translations of the elliptic curve. The integral Ω is (up to a trivial normalization factor) a standard elliptic integral, it is given by an hypergeometric function in the variable

$$m := \frac{s_k(s_l - s_m)}{s_l(s_k - s_m)} \tag{7.10}$$

or a modular form in terms of q.

The differential of R is given on $F_{\mathbf{u}}$ by $dR = \, J(Z)\, \omega$ where

$$J(Z) = \, 2\,(s_m - s_l)\, c_k\, t_k\, \frac{Z_0\, Z_1\, Z_2\, Z_3}{(Z_m^2 + \, c_k\, Z_l^2)^2} \tag{7.11}$$

The period Ω does not vanish and $J(Z)$, $Z \in F_{\mathbf{u}}(0)$, only vanishes on the 4 "ramification points" necessarily present due to the symmetries.

Corollary 16 *The Jacobian of the map θ^t is given by the equality*

$$\theta_*(ch_{\frac{3}{2}}(U)) = 3\,\Omega\, J\, V$$

where J is the element of the center $C^\infty(F_{\mathbf{u}}(0))$ of $C^\infty(F_{\mathbf{u}} \times_{\sigma,\mathcal{L}} \mathbb{Z})$ given by formula (7.11).

This statement assumes that $\mathbf{u}$ is generic in the measure theoretic sense so that η admits good diophantine approximation ([5]). It justifies in particular the terminology of "ramified covering" applied to θ^t. The function J has only 4 zeros on $F_{\mathbf{u}}(0)$ which correspond to the ramification.
As shown by Theorem 5 the algebra $\mathcal{A}_{\mathbf{u}}$ is defined over $\mathbb{R}$, i.e. admits a natural antilinear automorphism of period two, γ uniquely defined by

$$\gamma(Y_\mu) := Y_\mu\,, \quad \forall \mu \tag{7.12}$$

Theorem 5 also shows that σ is defined over $\mathbb{R}$ and hence commutes with complex conjugation $c(Z) = \bar{Z}$. This gives a natural real structure γ on the algebra C_Q with $C = F_{\mathbf{u}} \times F_{\mathbf{u}}$ and Q as above,

$$\gamma(f(Z, Z')) := \overline{f(c(Z), c(Z'))}\,, \quad \gamma(W_L) := W_{c(L)}\,, \quad \gamma(W'_{L'}) := W'_{c(L')}$$

One checks that the morphism ρ of lemma 7 is "real" i.e. that,

$$\gamma \circ \rho = \rho \circ \gamma \tag{7.13}$$

Since $c(Z) = \bar{Z}$ reverses the orientation of $F_{\mathbf{u}}$, while γ preserves the orientation of $S^3_{\mathbf{u}}$ it follows that $J(\bar{Z}) = -J(Z)$ and J necessarily vanishes on $F_{\mathbf{u}}(0) \cap P_3(\mathbb{R})$.
Note also that for general h one has $\langle \mathrm{ch}_{\frac{3}{2}}(U), \tau_h \rangle \neq 0$ which shows that both $\mathrm{ch}_{\frac{3}{2}}(U) \in HH_3$ and $\tau_h \in HH^3$ are non trivial Hochschild classes. These results hold in the smooth algebra $C^\infty(S^3_{\mathbf{u}})$ containing the closure of $C_{\mathrm{alg}}(S^3_{\mathbf{u}})$ under holomorphic functional calculus in the C^* algebra $C^*(S^3_{\mathbf{u}})$. We can also use Theorem 15 to show the non-triviality of the morphism $\theta_v : C_{\mathrm{alg}}(S^3_{\mathbf{u}}) \mapsto C^\infty(\mathbb{T}^2_\eta)$ of (6.21).

Corollary 17 *The pullback of the fundamental class $[\mathbb{T}^2_\eta]$ of the noncommutative torus by the homomorphism $\theta_v : C_{\mathrm{alg}}(S^3_{\mathbf{u}}) \mapsto C^\infty(\mathbb{T}^2_\eta)$ of (6.21) is non zero, $\theta_v^*([\mathbb{T}^2_\eta]) \neq 0 \in HH^2$ provided v is not a ramification point.*

We have shown above the non-triviality of the Hochschild homology and cohomology groups $HH_3(C^\infty(S^3_{\mathbf{u}}))$ and $HH^3(C^\infty(S^3_{\mathbf{u}}))$ by exhibiting specific elements with non-zero pairing. Combining the ramified cover $\pi = \theta^t$ with the natural spectral geometry (spectral triple) on the noncommutative 3-dimensional nilmanifold $F_{\mathbf{u}} \times_{\sigma,\mathcal{L}} \mathbb{Z}$ yields a natural spectral triple on $S^3_{\mathbf{u}}$ in the generic case. It will be analysed in Part II, together with the C^*-algebra $C^*(S^3_{\mathbf{u}})$, the vanishing of the primary class of U in K_1, and the cyclic cohomology of $C^\infty(S^3_{\mathbf{u}})$.

8 Calculus and Cyclic Cohomology

Theorem 15 suggests the existence of a “rational" form of the calculus explaining the appearance of the elliptic period Ω and the rationality of R. We shall show in this last section that this indeed the case.
Let us first go back to the general framework of twisted cross products of the form

$$\mathcal{A} = C^\infty(M) \times_{\sigma,\mathcal{L}} \mathbb{Z} \tag{8.1}$$

where σ is a diffeomorphism of the manifold M. We shall follow [6] to construct cyclic cohomology classes from cocycles in the bicomplex of group cohomology (with group $\Gamma = \mathbb{Z}$) with coefficients in de Rham currents on M. The twist by the line bundle $\mathcal{L}$ introduces a non-trivial interesting nuance.
Let $\Omega(M)$ be the algebra of smooth differential forms on M, endowed with the action of $\mathbb{Z}$

$$\alpha_{1,k}(\omega) := \sigma^{*k}\omega\,, \quad k \in \mathbb{Z} \tag{8.2}$$

As in [8] p. 219 we let $\tilde{\Omega}(M)$ be the graded algebra obtained as the (graded) tensor product of $\Omega(M)$ by the exterior algebra $\wedge(\mathbb{C}[\mathbb{Z}]')$ on the augmentation ideal $\mathbb{C}[\mathbb{Z}]'$ in the group ring $\mathbb{C}[\mathbb{Z}]$. With $[n], n \in \mathbb{Z}$ the canonical basis of

$\mathbb{C}[\mathbb{Z}]$, the augmentation $\epsilon : \mathbb{C}[\mathbb{Z}] \mapsto \mathbb{C}$ fulfills $\epsilon([n]) = 1, \forall n$, and

$$\delta_n := [n] - [0] , \quad n \in \mathbb{Z} , \quad n \neq 0 \tag{8.3}$$

is a linear basis of $\mathbb{C}[\mathbb{Z}]'$. The left regular representation of $\mathbb{Z}$ on $\mathbb{C}[\mathbb{Z}]$ restricts to $\mathbb{C}[\mathbb{Z}]'$ and is given on the above basis by

$$\alpha_{2,k}(\delta_n) := \delta_{n+k} - \delta_k , \quad k \in \mathbb{Z} \tag{8.4}$$

It extends to an action α_2 of $\mathbb{Z}$ by automorphisms of $\wedge\mathbb{C}[\mathbb{Z}]'$. We let $\alpha = \alpha_1 \otimes \alpha_2$ be the tensor product action of $\mathbb{Z}$ on $\tilde{\Omega}(M) = \Omega(M) \otimes \wedge\mathbb{C}[\mathbb{Z}]'$.
We now use the hermitian line bundle $\mathcal{L}$ to form the twisted cross-product

$$\mathcal{C} := \tilde{\Omega}(M) \times_{\alpha\,,\mathcal{L}} \mathbb{Z} \tag{8.5}$$

We let $\mathcal{L}_n$ be as in (6.5) for $n > 0$ and extend its definition for $n < 0$ so that $\mathcal{L}_{-n}$ is the pullback by σ^n of the dual $\hat{\mathcal{L}}_n$ of $\mathcal{L}_n$ for all n. The hermitian structure gives an antilinear isomorphism $* : \mathcal{L}_n \mapsto \hat{\mathcal{L}}_n$. The algebra $\mathcal{C}$ is the linear span of monomials $\xi\, W^n$ where

$$\xi \in C^\infty(M, \mathcal{L}_n) \otimes_{C^\infty(M)} \tilde{\Omega}(M) \tag{8.6}$$

with the product rules (6.7), (6.8).
Let ∇ be a hermitian connection on $\mathcal{L}$. We shall turn $\mathcal{C}$ into a differential graded algebra. By functoriality ∇ gives a hermitian connection on the $\mathcal{L}_k$ and hence a graded derivation

$$\nabla_n : C^\infty(M, \mathcal{L}_n) \otimes_{C^\infty(M)} \Omega(M) \mapsto C^\infty(M, \mathcal{L}_n) \otimes_{C^\infty(M)} \Omega(M) \tag{8.7}$$

whose square ∇_n^2 is multiplication by the curvature $\kappa_n \in \Omega^2(M)$ of $\mathcal{L}_n$,

$$\kappa_{n+m} = \kappa_n + \sigma^{*n}(\kappa_m) , \quad \forall n,\, m \in \mathbb{Z} \tag{8.8}$$

with $\kappa_1 = \kappa \in \Omega^2(M)$ the curvature of $\mathcal{L}$. Ones has $d\kappa_n = 0$ and we extend the differential d to a graded derivation on $\tilde{\Omega}(M)$ by

$$d\delta_n = \kappa_n \tag{8.9}$$

We can then extend ∇_n uniquely to

$$\tilde{\nabla}_n : C^\infty(M, \mathcal{L}_n) \otimes_{C^\infty(M)} \tilde{\Omega}(M) \mapsto C^\infty(M, \mathcal{L}_n) \otimes_{C^\infty(M)} \tilde{\Omega}(M) \tag{8.10}$$

so that it fulfills

$$\tilde{\nabla}_n(\xi\,\omega) = \tilde{\nabla}_n(\xi)\,\omega + (-1)^{\deg(\xi)}\xi\, d\omega\,, \quad \forall \omega \in \tilde{\Omega}(M) \tag{8.11}$$

Proposition 18 *(i) The graded derivation d of $\tilde{\Omega}(M)$ extends uniquely to a graded derivation of $\mathcal{C}$ such that,*

$$d(\xi\, W^n) := (\tilde{\nabla}_n(\xi) - (-1)^{\deg(\xi)}\xi\, \delta_n)\, W^n$$

(ii) The pair $(\mathcal{C},\, d)$ is a graded differential algebra.

To construct closed graded traces on this differential graded algebra we follow ([6]) and consider the double complex of group cochains (with group $\Gamma = \mathbb{Z}$) with coefficients in de Rham currents on M. The cochains $\gamma \in C^{n,m}$ are totally antisymmetric maps from $\mathbb{Z}^{n+1}$ to the space $\Omega_{-m}(M)$ of de Rham currents of dimension $-m$, which fulfill

$$\gamma(k_0 + k, k_1 + k, k_2 + k, \cdots, k_n + k) = \sigma_*^{-k}\gamma(k_0, k_1, k_2, \cdots, k_n)\,, \quad \forall k,\ k_j \in \mathbb{Z}$$

Besides the coboundary d_1 of group cohomology, given by

$$(d_1\gamma)(k_0, k_1, \cdots, k_{n+1}) = \sum_0^{n+1} (-1)^{j+m}\, \gamma(k_0, k_1, \cdots, \hat{k_j}, \cdots, k_{n+1})$$

and the coboundary d_2 of de Rham homology,

$$(d_2\gamma)(k_0, k_1, \cdots, k_n) = b(\gamma(k_0, k_1, \cdots, k_n))$$

the curvatures κ_n generate the further coboundary d_3 defined on Ker d_1 by,

$$(d_3\gamma)(k_0, \cdots, k_{n+1}) = \sum_0^{n+1} (-1)^{j+m+1}\, \kappa_{k_j} \gamma(k_0, \cdots, \hat{k_j}, \cdots, k_{n+1}) \tag{8.12}$$

which maps Ker $d_1 \cap C^{n,m}$ to $C^{n+1,m+2}$. Translation invariance follows from (8.8) and $\varphi_*(\omega C) = \varphi^{*-1}(\omega)\varphi_*(C)$ for $C \in \Omega_{-m}(M)$, $\omega \in \Omega^*(M)$.
To each $\gamma \in C^{n,m}$ one associates the functional $\tilde{\gamma}$ on $\mathcal{C}$ given by,

$$\begin{aligned} \tilde{\gamma}(\,\xi\, W^n) &= 0\,, \quad \forall\, n \neq 0\,, \quad \xi \in \tilde{\Omega}(M) \\ \tilde{\gamma}(\,\omega \otimes \delta_{k_1} \cdots \delta_{k_n}) &= \langle \omega, \gamma(0, k_1 \cdots, k_n)\rangle\,, \quad \forall k_j \in \mathbb{Z} \end{aligned} \tag{8.13}$$

and the $(n-m+1)$ linear form on $\mathcal{A} = C^\infty(M) \times_{\sigma, \mathcal{L}} \mathbb{Z}$ given by,

$$\begin{aligned} &\Phi(\gamma)(a_0, a_1, \cdots, a_{n-m}) = \\ &\lambda_{n,m} \sum_0^{n-m} (-1)^{j(n-m-j)}\, \tilde{\gamma}(da_{j+1} \cdots da_{n-m}\, a_0\, da_1 \cdots da_{j-1}\, da_j) \end{aligned} \tag{8.14}$$

where $\lambda_{n,m} := \frac{n!}{(n-m+1)!}$.

Lemma 19 *(i) The Hochschild coboundary $b\Phi(\gamma)$ is equal to $\Phi(d_1\gamma)$.*
(ii) Let $\gamma \in C^{n,m} \cap$ Ker d_1. Then $\Phi(\gamma)$ is a Hochschild cocycle and

$$B\Phi(\gamma) = \Phi(d_2\gamma) + \tfrac{1}{n+1}\, \Phi(d_3\gamma)$$

We shall now show how the above general framework allows to reformulate the calculus involved in Theorem 15 in rational terms. We let M be the elliptic curve $F_{\mathbf{u}}$ where $\mathbf{u}$ is generic and even. Let then ∇ be an arbitrary hermitian connection on $\mathcal{L}$ and κ its curvature. We first display a cocycle $\gamma = \sum \gamma_{n,m} \in \sum C^{n,m}$ which reproduces the cyclic cocycle τ_3.

Lemma 20 *There exists a two form α on $M = F_{\mathbf{u}}$ and a multiple $\lambda\, dv$ of the translation invariant two form dv such that :*

(i) $$\kappa_n = n\,\lambda\, dv + (\sigma^{*n}\alpha - \alpha)\,, \quad \forall n \in \mathbb{Z}$$

(ii) $$d_2(\gamma_j) = 0\,, \quad d_1(\gamma_3) = 0\,, \quad d_1(\gamma_1) + \tfrac{1}{2}\, d_3(\gamma_3) = 0\,, \quad B\Phi(\gamma_1) = 0\,,$$

where $\gamma_1 \in C^{1,0}$ and $\gamma_3 \in C^{1,-2}$ are given by

$$\gamma_1(k_0, k_1) := \tfrac{1}{2}\,(k_1 - k_0)(\sigma^{*k_0}\alpha + \sigma^{*k_1}\alpha)\,, \quad \gamma_3(k_0, k_1) := k_1 - k_0\,, \quad \forall k_j \in \mathbb{Z}$$

(iii) The class of the cyclic cocycle $\Phi(\gamma_1) + \Phi(\gamma_3)$ is equal to τ_3.

We use the generic hypothesis in the measure theoretic sense to solve the "small denominator" problem in (i). In (ii) we identify differential forms $\omega \in \Omega^d$ of degree d with the dual currents of dimension $2 - d$.

It is a general principle explained in [5] that a cyclic cocycle τ generates a calculus whose differential graded algebra is obtained as the quotient of the universal one by the radical of τ. We shall now explicitely describe the reduced calculus obtained from the cocycle of lemma 20 *(iii)*. We use as above the hermitian line bundle $\mathcal{L}$ to form the twisted cross-product

$$\mathcal{B} := \Omega(M) \times_{\alpha,\mathcal{L}} \mathbb{Z} \tag{8.15}$$

of the algebra $\Omega(M)$ of differential forms on M by the diffeomorphism σ. Instead of having to adjoin the infinite number of odd elements δ_n we just adjoin two χ and X as follows. We let δ be the derivation of $\mathcal{B}$ such that

$$\delta(\xi\, W^n) := i\, n\, \xi\, W^n\,, \quad \forall \xi \in C^\infty(M, \mathcal{L}_n) \otimes_{C^\infty(M)} \Omega(M) \tag{8.16}$$

We adjoin χ to $\mathcal{B}$ by tensoring $\mathcal{B}$ with the exterior algebra $\wedge\{\chi\}$ generated by an element χ of degree 1, and extend the connection ∇ (8.7) to the unique

graded derivation d' of $\Omega' = \mathcal{B} \otimes \wedge\{\chi\}$ such that,

$$\begin{aligned} d'\,\omega &= \nabla\omega + \chi\delta(\omega)\,, \quad \forall \omega \in \mathcal{B} \\ d'\chi &= -\,\lambda\, dv \end{aligned} \tag{8.17}$$

with $\lambda\, dv$ as in lemma 20. By construction, every element of Ω' is of the form

$$y = b_0 + b_1\,\chi\,, \quad b_j \in \mathcal{B} \tag{8.18}$$

One does not yet have a graded differential algebra since $d'^2 \neq 0$. However, with α as in lemma 20 one has

$$d'^2(x) = [\,x,\,\alpha]\,, \quad \forall x \in \Omega' = \mathcal{B} \otimes \wedge\{\chi\} \tag{8.19}$$

and one can apply lemma 9 p.229 of [8] to get a differential graded algebra by adjoining the degree 1 element $X :=$ "$d1$" fulfilling the rules

$$X^2 = -\alpha\,, \quad x\,X\,y = 0\,, \quad \forall x, y \in \Omega' \tag{8.20}$$

and defining the differential d by,

$$\begin{aligned} d\,x &= d'\,x + [\,X,\,x]\,, \quad \forall x \in \Omega' \\ dX &= 0 \end{aligned} \tag{8.21}$$

where $[\,X,\,x]$ is the graded commutator. It follows from lemma 9 p.229 of [8] that we obtain a differential graded algebra Ω^*, generated by $\mathcal{B}$, ξ and X. In fact using (8.20) every element of Ω^* is of the form

$$x = x_{1,1} + x_{1,2}\,X + X\,x_{2,1} + X\,x_{2,2}X\,, \quad x_{i,j} \in \Omega' \tag{8.22}$$

and we define the functional $\int$ on Ω^* by extending the ordinary integral,

$$\int \omega := \int_M \omega\,, \quad \forall \omega \in \Omega(M) \tag{8.23}$$

first to $\mathcal{B} := \Omega(M) \times_{\alpha,\mathcal{L}} \mathbb{Z}$ by

$$\int \xi\, W^n := 0\,, \quad \forall n \neq 0 \tag{8.24}$$

then to Ω' by

$$\int (b_0 + b_1\,\chi) := \int b_1\,, \quad \forall b_j \in \mathcal{B} \tag{8.25}$$

and finally to Ω^* as in lemma 9 p.229 of [8],

$$\int (x_{1,1} + x_{1,2}\,X + X\,x_{2,1} + X\,x_{2,2}X) := \int x_{1,1} + (-1)^{\deg(x_{2,2})} \int x_{2,2}\,\alpha \tag{8.26}$$

Theorem 21 *Let $M = F_{\mathbf{u}}$, ∇, α be as in lemma 20.*
The algebra Ω^* is a differential graded algebra containing $C^\infty(M) \times_{\alpha,\mathcal{L}} \mathbb{Z}$.
The functional $\int$ is a closed graded trace on Ω^*.
The character of the corresponding cycle on $C^\infty(M) \times_{\alpha,\mathcal{L}} \mathbb{Z}$

$$\tau(a_0, \cdots, a_3) := \int a_0\, da_1 \cdots da_3\,, \quad \forall a_j \in C^\infty(M) \times_{\alpha,\mathcal{L}} \mathbb{Z}$$

is cohomologous to the cyclic cocycle τ_3.

It is worth noticing that the above calculus fits with [4], [12], and [11].
Now in our case the line bundle $\mathcal{L}$ is holomorphic and we can apply Theorem 21 to its canonical hermitian connection ∇. We take the notations of section 5, with $C = F_{\mathbf{u}} \times F_{\mathbf{u}}$, and Q given by (6.20). This gives a particular "rational" form of the calculus which explains the rationality of the answer in Theorem 15. We first extend as follows the construction of C_Q. We let $\Omega(C, Q)$ be the generalised cross-product of the algebra $\Omega(C)$ of meromorphic differential forms (in dZ and dZ') on C by the transformation $\tilde{\sigma}$. The generators W_L and $W'_{L'}$ fulfill the cross-product rules,

$$W_L\,\omega = \tilde{\sigma}^*(\omega)\,W_L\,, \qquad W'_{L'}\,\omega = (\tilde{\sigma}^{-1})^*(\omega)\,W'_{L'} \tag{8.27}$$

while (5.3) is unchanged. The connection ∇ is the restriction to the subspace $\{Z' = \bar{Z}\}$ of the unique graded derivation ∇ on $\Omega(C, Q)$ which induces the usual differential on $\Omega(C)$ and satisfies,

$$\begin{aligned} \nabla W_L &= (d_Z \log L(Z) - d_Z \log Q(Z, Z'))\, W_L \\ \nabla W'_{L'} &= W'_{L'}\, (d_{Z'} \log L'(Z') - d_{Z'} \log Q(Z, Z')) \end{aligned} \tag{8.28}$$

where d_Z and $d_{Z'}$ are the (partial) differentials relative to the variables Z and Z'. Note that one needs to check that the involved differential forms such as $d_Z \log L(Z) - d_Z \log Q(Z, Z')$ are not only invariant under the scaling transformations $Z \mapsto \lambda Z$ but are also basic, i.e. have zero restriction to the fibers of the map $\mathbb{C}^4 \mapsto P_3(\mathbb{C})$, in both variables Z and Z'. By definition the derivation $\delta_\kappa = \nabla^2$ of $\Omega(C, Q)$ vanishes on $\Omega(C)$ and fulfills

$$\delta_\kappa(W_L) = \kappa\, W_L\,, \qquad \delta_\kappa(W'_{L'}) = -\, W'_{L'}\, \kappa \tag{8.29}$$

where

$$\kappa = d_Z\, d_{Z'} \log Q(Z, Z') \tag{8.30}$$

is a basic form which when restricted to the subspace $\{Z' = \bar{Z}\}$ is the curvature. We let as above δ be the derivation of $\Omega(C, Q)$ which vanishes on $\Omega(C)$ and is such that $\delta W_L = i\, W_L$ and $\delta W'_{L'} = -i\, W'_{L'}$. We proceed exactly as above and get the graded algebras $\Omega' = \Omega(C, Q) \otimes \wedge\{\chi\}$ obtained by adjoining χ and Ω^* by adjoining X. We define d', d as in (8.17) and (8.21) and the integral $\int$ by integration (8.23) on the subspace $\{Z' = \bar{Z}\}$ followed as above by steps (8.24), (8.25), (8.26).

Corollary 22 *Let ρ: $C_{\mathrm{alg}}(S^3_{\mathbf{u}}) \mapsto C_Q$ be the morphism of lemma 7. The equality*

$$\tau_{\mathrm{alg}}(a_0, \cdots, a_3) := \int \rho(a_0)\, d'\rho(a_1) \cdots d'\rho(a_3)$$

defines a 3-dimensional Hochschild cocycle $\tau_{\rm alg}$ *on* $C_{\rm alg}(S^3_{\mathbf{u}})$.
Let $h \in$ *Center* $(C^\infty(F_{\mathbf{u}} \times_{\sigma,\mathcal{L}} \mathbb{Z})) \sim C^\infty(F_{\mathbf{u}}(0))$. *Then*

$$\langle ch_{\frac{3}{2}}(U), \tau_h \rangle = \langle h\, ch_{\frac{3}{2}}(U), \tau_{\rm alg} \rangle$$

The computation of d' only involves rational fractions in the variables Z, Z' (8.28), and the formula (7.5) for $\mathrm{ch}_{\frac{3}{2}}(U)$ is polynomial in the W_L, $W'_{L'}$. We thus obtain an a priori reason for the rational form of the result of Theorem 15. The explicit computation as well as its extension to the odd case and the degenerate cases will be described in part II.

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